RESERVE BANK
OF NEW ZEALAND

DP2009/??

Impulse Response Identification in DSGE Models

Martin Fukač

June 2009

JEL classification: C30, C52

www.rbnz.govt.nz/research/discusspapers/

Discussion Paper Series

ISSN 1177-7567

DP2009/??

Impulse Response Identification in DSGE Models*

Martin Fukač†

Abstract

I apply the global identification theory developed by by Ramírez, Waggoner and Zha (2007) for SVAR models to invertible dynamic stochastic general equilibrium (DSGE) models. I address the question whether unobserved exogenous shocks in such DSGE models can be uniquely estimated from available data. As the location of shocks matters for the identification, the methodology can be a useful tool in the development process of DSGE models, to help to discriminate among model structures. I demonstrate the use of the methodology on a basic New Keynesian business cycle model.

* I am indebted to Jaromír Beneš for many provocative and discerning discussions on this topic. This paper was also greatly influenced by numerous conversations with Tao Zha. I also thank the participants of the research seminars at RBNZ, 13th Australasian Macroeconomics Workshop in Sydney, and of the Phillips Symposium in Wellington in 2008 for their comments.

† Economics Department, 2 The Terrace, Wellington, New Zealand. E-mail: *firstname.lastname*@rbnz.govt.nz.
ISSN 1177-7567

Contents

1	Introduction	2
2	The identification problem	4
3	Identification methodology	6
	3.1 Inverting a DSGE model	6
	3.2 Impulse response identification	7
	3.3 Checking minimal system realisation	10
4	An application	13
5	Conclusion	19
6	References	19
A	Proof of Theorem 1	21
B	Proof of Theorem 2	22
C	Proof of Lemma 1	22
D	Solution to examples in Section 3	23
E	Matrix pseudoinverse	27

1 Introduction

I advance the idea that the identifiability of impuls responses should be one of the key criteria during the development phase of dynamic stochastic general equilibrium (DSGE) models in central banks, in particular if such models are intended to be core models to produce central forecasts. In my view, only a limited attention has been devoted to the problem of impulse response identification as a key to unique story-telling forecast. In this paper, I show how an existing methodology for structural vector autoregressions (SVAR) developed in Rubio-Ramirez, Waggoner, and Zha (2008) can be adopted for the identification of invertible DSGE models.

There is a growing interest among central banks to employ (DSGE) models as their central or secondary forecasting models. The success of their application depends on models' ability to credibly replicate and interpret key economic data. In the central banks' slang this is often referred to as producing *story-telling forecasts*, and draw inferences for the future. Some DSGE models are calibrated, but the credibility of the story embeded in such a model structure is enhanced if it is estimated. Consistent estimation depends, however, on the ability to uniquely inform model parameters by the data (structural identification).

In this paper, I analyze the structural identification problem from the perspective of structural shocks estimates, and model impulse responses. This takes us only a half way towards full DSGE model identification, but it is sufficiently far to find out if an estimated DSGE model has (i) a unique estimate of initial conditions (like position in the business cycle), which are the result of the cummulative effect of all past shocks the model economy faced, and (ii) unique impulse responses to current structural shocks. This is very useful for practical purposes as (i) guarantees consistent and unique interpretation of historical data, and (ii) guarantees unique model forecasts. The SVAR identification methodology by Rubio-Ramirez et al. (2009) provides a neccessary condition for global identification, which is elegant and easy to apply to invertible DSGE models.

The methodology, I propose here, consists of 3 steps. After a log-linearisation, the model is solved for rational expectations and represented in its state-space form (the core model dynamics is captured by the system of state equations. The states – model endogenous variables, are mapped to observable data series via the system of measurement equations). The first step is to invert the state-space model in to a structural VAR model. You will see that it is only possible the number of observable variables is the same as the number of shocks. I call the inverted state-space model as the *semi-structural model* here. The second step is the application of SVAR identification theory by Rubio- Ramirez et al. (2009), which provides a necessary condition for global identification. The third step is to check if the state-space model is of a minimal realisation. I believe that it is desirable that a policy-oriented model exhibits such a property. If a system is minimal it meanns that it is both controllable and observable. Observability guarantees that all the model endogenous (state) variables can be uniquely recovered from observed data. And controllability guarantees that unique exogenous shocks can be recovered from the state variables. If the state-space model is minimal, then if the model is estimated we can identify unique initial conditions for the given set of observable data and model. In the current DSGE literature, it seems that this property is omitted, or does not play a central role. But minimal state-space system realisation is important for policy oriented models and forecasting models, in particular, if one's aim is a coherent and unique structural story, and the provision of credible policy recommendations.

In the rest of the paper I describe the methodology in details. The problem of impulse response identification is discussed in the next section. In the third section I propose how to apply the SVAR identification approach to DSGE models. As a byproduct, I show how Rubio, Waggoner, and Zha's (2007) approach can be easily extended to models with more shocks than observable variables. In the fourth section I demonstarte this paper's methodology on an example of a simple New Keynesian business cycle model.

2 The identification problem

A typical DSGE model is non-linear and forward-looking expectations are its distinctive feature:

$$0 = \Theta(E_t x_{t+1}, x_t, x_{t-1}, u_t). \tag{1}$$

x_t is $(r \times 1)$ vector of model variables; u_t is the $(k \times 1)$ vector of structural shocks, where $k \leq r$. The shocks are uncorrelated iid $N(0, \sigma_{u_i}^2)$ for $i = 1, ..., k$. Θ is a non-linear function relating the endogenous and exogenous variables using a set of *deep-structural parameters* θ. θs captures microeconomic characteristics of the economic agents in the model (like their time preferences, risk aversion, frequency of price adjustment, retained earnings, tax rates or inflation target).

I shall work with the (log)linearised form of the DSGE model (1), and call it the *structural model*:

$$B_0 x_t = B_1 E_t x_{t+1} + B_2 x_{t-1} + F u_t, \tag{2}$$

B_0, B_1, $B_2 \in \mathbb{R}^{r \times r}$ $(r \times r)$ and $F \in \mathbb{R}^{r \times k}$ are full column rank matrices. The elements of these matrices are functions of the deep-structural parameters θ.

For simplicity at this stage, I assume that all model variables are measurable. Solving the structural model for the rational expectations (e.g., by the method of undetermined coeficients), I obtain the *semi-structural model*:[1]

$$G_0 x_t = G_1 x_{t-1} + u_t.$$

The semi-structural form is the object of interest here. $G_0 \in \mathbb{R}^{r \times r}$ defines the contemporaneous relations among the endogenous variables, and $G_1 \in \mathbb{R}^{r \times r}$ captures their dynamics. The elements of the Gs, which I denote as η, are functions of the deep-structural parameters θ: $\eta = \eta(\theta)$. Note that the semi-structural model is nothing else but a finite order SVAR model, where ηs determine the responses of the model to exogenous shocks.

The central question of this paper is: Are the restrictions on G_0 such that impulse responses are identified? An impulse response is identified if ηs can be uniquely estimated from available data.

[1] I use the term "semi" because the model does not explicitly contain the forward-looking terms. As the rational expectations are model consistent, they are still present implicitly.

Definition 1. *The impulse responses of (2) are* globally identified *if the set of G_0 and G_1's elements $\eta \in R$ is not observationally equivalent with another set $\tilde{\eta} \in R$. The two sets are observationally equivalent if $L(\eta) = L(\tilde{\eta})$, where $L(.)$ is a likelihood function.*

Definition 2. *The impulse responses of (2) are* locally identified *if there exists some neighborhoud B in which the set of G_0 and G_1's elements $\eta \in R$ is not observationally equivalent to another set $\tilde{\eta} \in R \cap B$.*

When we estimate a DSGE model, we estimate the deep structural parameters θ, so why does it make sense to look at the identifiability of ηs rather than θs? Fukač, Pagan, and Pavlov (2007) touch this question. Unique θs are key for policy experiments and welfare analysis, but if the model is used for forecasting, ηs is the sufficient object of interest. Lets use the prisim of the Fisher information matrix from Rothenberg (1971), where the general conditions for the global identifiability are laid down. The Fisher information matrix for the set of deep structural parameters θ, is given by the variance of the scores: $E\left[\frac{\partial L(\theta)}{\partial \theta}\right]^2$. $L(.)$ is the likelihood function. By the chain rule the likelihood gradient can be decomposed as $\frac{\partial L(\theta)}{\partial \theta} = \frac{\partial L(\theta)}{\partial \eta}\frac{\partial \eta}{\partial \theta}$. The information for θ will be the Fisher information for η times the square of $\frac{\partial \eta}{\partial \theta}$. If the latter is singular then the information matrix for θ is singular, which indicates that some of the parameters in θ are not identified. That is what all standard identification checks would find. But note that the singularity can appear even though that for η may not. Even though looking at the identifiability of θs is natural, it is an unnecessarily strict (and also irelevant) prism for the purposes of impulse responses identification. Even though θ lacks identification, the model can generate unique dynamics because η can be identified.

The identification problem for the semi-structural model is in principle the same as for a structural VAR model. The major difference is that the DSGE model may contain latent variables, in which the problem of DSGE model invertibility arises. It depends on the number of model variables (how many of them we can statistically measure), and on the number of exogenous shocks. In the next section we will see that the dimension of shocks is key.

3 Identification methodology

The model (2) has the minimum state variable (MSV) solution of the form

$$G_0 x_t = G_1 x_{t-1} + Q u_t, \tag{3}$$

where G_0 and G_1 are $(r \times r)$ matrices, and Q is a $(r \times k)$, full column rank matrix. $G_0 = B_0 - B_1 G_0^{-1} G_1$, $G_1 = B_2$, and $Q = F$.

For estimation the Kalman filter is used, and model (3) is put into a state-space form. The MSV solution establishes the transition equation:

$$x_t = \mathbf{A} x_{t-1} + \mathbf{B} u_t. \tag{4}$$

$\mathbf{A} = G_0^{-1} G_1$, and $\mathbf{B} = G_0^{-1} F$. The map of the state (model endogenous) variables to their observable counterparts establishes the measurement equation

$$y_t = C x_t. \tag{5}$$

y_t is the $(n \times 1)$ vector of observable variables. C is a $(n \times r)$ matrix, and $r \geq n$. For simplicity, I do not assume any measurement errors in (5) but the reader is invited to check that the derivations below also hold in that case.

3.1 Inverting a DSGE model

To be able to apply the methodology by Rubio-Ramirez et al. (2009) and identify impulse responses, the model (4)-(5) has to be written in terms of endogenous observable variables and their own past values. Inverting the state-space system in such a form yields a SVAR model type structure, which I am going to call the *semi-structural model* here. In the engineering literature, where such an inversion comes from, is called the impulse response function (see Ljung, 1999, Section 4.3). Villaverde *et al.* (2007) study the properties of such transformation for economic problems.

In general, there are more state variables than observable variables[2]. As the first step in deriving the semi-structural form, I substitute (4) into (5) and rewrite the measurement equation as

$$y_t = \mathbf{C}x_{t-1} + \mathbf{D}u_t, \tag{6}$$

where $\mathbf{C} = C\mathbf{A}$ and $\mathbf{D} = C\mathbf{B}$. $\mathbf{D}$ might not be invertible (or left invertible), because, it does not necessary have a full column rank, which is key in rewriting the model in terms of observable variables only. I have to impose the following assumption.

Assumption 1. *$\boldsymbol{D}$ is invertible, or at least left invertible, ie. $\boldsymbol{D}^+\boldsymbol{D} = I$.*

This assumption restricts us to the state-space models that have the same number of shocks (structural and/or measurement errors) as observable variables.

As the next step, I use Assumption 1 and solve for u_t from (6) by taking the left inverse of $\mathbf{D}$. Then $u_t = \mathbf{D}^+ y_t - \mathbf{D}^+\mathbf{C}x_{t-1}$. I plug this expression into (4), and re-arrange to get $x_t = [\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{D}^+\mathbf{C})L]^{-1}\mathbf{B}\mathbf{D}^+ y_t$. Substituting that back to the measurement equation (6), we get a SVAR(∞) representation of our DSGE model

$$\mathbf{D}^+ y_t = \mathbf{D}^+\mathbf{C}\sum_{j=0}^{\infty}(\mathbf{A} - \mathbf{B}\mathbf{D}^+\mathbf{C})^j\mathbf{B}\mathbf{D}^+ y_{t-j-1} + u_t. \tag{7}$$

3.2 Impulse response identification

Before mapping the identification theory of SVAR models by Rubio-Ramirez, Waggoner, and Zha (2007; RWZ) in to the DSGE model problem, for the readers'

[2] This creates only a minor complication for the invertibility technique itself. If C is invertible ($k = n$), then it is straightforward to solve for the semi-reduced form. From the state equation (4), I express x_t and substitute it to the measurement equation (5).

$$\begin{aligned}
(I - \mathbf{A}L)x_t &= \mathbf{B}u_t \\
(I - \mathbf{A}L)C^{-1}y_t &= \mathbf{B}u_t \\
\mathbf{B}^+C^{-1}y_t &= \mathbf{B}^+\mathbf{A}C^{-1}y_{t-1} + u_t \\
A_0 y_t &= A_1 y_{t-1} + u_t
\end{aligned}$$

with $A_0 = \mathbf{B}^+C^{-1}$ and $A_1 = \mathbf{B}^+\mathbf{A}C^{-1}$. If $\mathbf{A}$ is a stable matrix – which is almost always, because it comes from the rational expectations solution – the state-space model can be represented as an structural VAR(1).

convenience, let me re-state here the key features of the theory. The next text follows the structure in RWZ (2007, Section II). I extend the theory to the situations where a SVAR representation has more shocks than observable variables.

To apply their methodology, I have to rewrite (7) as

$$y_t' A_0 = \mathbf{y}_t' \mathbf{A}_+(L) + u_t'. \tag{8}$$

$A_0 = (\mathbf{D}^+)' = [(CG_0^{-1}F)^{-1}]'$ is an $(n \times n)$ matrix capturing contemporaneous relationships among observed endogenous variables implied by the theoretical model. $\mathbf{y}_t = [y_{t-1} y_{t-2} \ldots y_{-\infty}]'$. $\mathbf{A}_+(L)' = [\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{D}^+\mathbf{C})L]^{-1}\mathbf{B}\mathbf{D}^+$ is an infinite polynomial. If $(\mathbf{A} - \mathbf{B}\mathbf{D}^+\mathbf{C})$ is stable then y_t is a bounded sequence. Please the footnote 2 that if $n = k = r$ and C is an identity matrix, then (8) shrinks to SVAR(1) – the MSV solution (3).

The adjustment For the sake of wider exposition, lets ommit Assumption 1 for a moment, and assume instead that A_0 is of the dimension $(k \times n)$. It is not an invertible matrix, but it has a full row rank. A reduced form representation is obtained by using the *pseudoinverse*.[3] As A_0 has a full row rank, pseudoinverse A_0^+ (dimension $(n \times k)$) is a right inverse matrix, i.e. $A_0 A_0^+ = I$, where I is an $(n \times n)$ identity matrix. The full row rank assumption must hold, in order to have n observable variables in the model. The reduced form is then

$$y_t' = \mathbf{y}_t' B + u_t',$$

where $B = A_+ A_0^+$ is of dimension $(m \times n)$, and $u_t' = \varepsilon_t' A_0^+$ is an $(1 \times n)$ vector of reduced structural shocks. Note that the dimension corresponds to the number of y_ts. We have as many reduced form shocks as observable variables. $E[u_t u_t'] = E[A_0^{+\prime} \varepsilon_t \varepsilon_t' A_0^+] = (A_0 A_0')^+ = \Sigma$, where Σ is an $(n \times n)$ variance-covariance matrix of reduced form shocks.

The first key theorem in Rubio-Ramirez et al. (2009) is about the observability equivalence.

[3] I recall the key computational rules with the pseudoinverse operator in Appendix E.

Theorem 1 (Observability equivalence)**.** *Two sets of structural parameters in (7),* (A_0, A_+) *and* $(\tilde{A_0}, \tilde{A_+})$*, are observationally equivalent if and only if there exists* $k \times k$ *orthogonal matrix* P *such that* $A_0 = \tilde{A}_0 P$ *and* $A_+ = \tilde{A}_+ P$*.*

Proof. see Appendix A. □

Representing restrictions For $1 \leq j \leq k$ and $f(A_0, A_0) = [A_0 \quad A_+]'$ of the dimension $(g \times k)$, where $g = n + m$, RWZ define matrix

$$M_j(f(A_0, A_+)) = \left[\begin{array}{cc} \multicolumn{2}{c}{Q_j f(A_0, A_+)} \\ \hline I_j & 0 \end{array} \right]$$

where I_j is an $(j \times j)$ identity matrix, and 0 is an $(j \times k - j)$ zero matrix. The linear restrictions can be represented by $(g \times g)$ matrices Q_j for $1 \leq j \leq k$. Each matrix Q_j has rank q_j. The structural parameters (A_0, A_+) satisfy the restrictions if and only if

$$Q_j f(A_0, A_+) e_j = 0,$$

where e_j is the j^{th} column of the $(k \times k)$ identity matrix I_k. The ordering of restrictions is important. The ordering of Q_j is such that

$$q_1 \geq q_2 \geq \ldots \geq q_k.$$

The following theorem is adjusted Theorem 5 from RWZ (2007).

Theorem 2 (The general rank condition)**.** *If* $(A_0, A_+) \in R$ *and* $M_j(f(A_0, A_+))$ *is of rank k for all* $1 \leq j \leq k$*, then the SVAR is globally identified at* (A_0, A_+)*.*

Proof. see Appendix B □

Checking identifiability Now we can return back to the impulse response identification in the DSGE case. Having the SVAR representation of the DSGE model, we can apply Theorem 2. Since A_+ is an infinite order polynomial, $f(A_0, A_+)$, is of an infinite size. But to apply the theorem, we can focus on a finite order model with $j = 1$, i.e. $A_2' = \mathbf{D}^+\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^+\mathbf{C})\mathbf{B}\mathbf{D}^+$, because the matrices

$A'_j = \mathbf{D}^+\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^+\mathbf{C})^j\mathbf{B}\mathbf{D}^+$ for $j > 1$ are combinations of A_2, and the rank of A_+ will be equal to the rank of A_2. Thus it is sufficient to construct $f(A_0,A_+)$ as

$$f(A_0,A_1,A_2) = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} [(\mathbf{D}^+]' \\ [\mathbf{D}^+\mathbf{C}\mathbf{B}\mathbf{D}^+]' \\ [\mathbf{D}^+\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^+\mathbf{C})\mathbf{B}\mathbf{D}^+]' \end{bmatrix}.$$

Given $f(A_0,A_1,A_2)$, we can form Q_j to represent zero restrictions, and correspondingly $M_j(f(A_0,A_+))$ for all $1 \le j \le k$.

Lemma 1. *If $\{(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^+\boldsymbol{C})^j\}_{j=3}^{\infty}$ is not an alternating series, then $f(A_0,A_1,A_2)$ captures characteristic exclusionary restrictions of (8).*

Proof. see Appendix C □

3.3 Checking minimal system realisation

I motivated this paper by the need of central banks to provide a credible interpretation of economic events in terms of structural shocks, and their effects on the current and future economic development. An essential part of a credible story is its consistency over time, as it is communicated to market participants on a regular basis. A typical central bank runs quarterly forecasting rounds, which result in an advice to policy makers. The estimation or judgement of the initial state of the economy is a typical challenge in each forecasting round. (i) What is the current position in the business cycle? (ii) What are the shocks that the economy has been facing and what will the economy's most likely response? This section discusses a necessary and sufficient conditions for state-space models, to get unique information on both (i) and (ii) above.

Having properly identified initial conditions is necessary for unique forecasts. The condition is met if the state-space representation is of a minimal realisation. The minimal realisation comes from the control theory and is related to model observability and controllability. Lets formally introduce these terms and explain what they mean in terms of DSGE models in a policy environment. I recapitulate the concepts of controllability and observability from the engeneering literature, The condition of minimal realisation may appear quite restrictive, as one may belive that we live in uncontrolable world, but from a perspective of a decision-maker, it is appealing to work with the model structures that satisfy such a property. She needs to have a fixed point from which she can disentangle the economic story. Uniquely estimated initial conditions provide such a fixed point. Furthermore, the uniqueness should also guarantee that the interpretation of the past development will not dramatically change and stays consistent over time. This is important for bank's communication policy.

Definition 3 (Observability)**.** *The state-space system $\{A,B,C,D\}$ is called observable if the observability matrix $O_n(C,A)$ has rank n.*

$$O_T(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix}$$

If the system is observable, then we can always solve for the initial state x_0 from given set of shocks u_t (typically being assumed to be zero), and observables y_t, for $t \geq 0$.

Definition 4 (Controllability)**.** *The state-system $\{A,B,C,D\}$ is called controllable if the controllability matrix $C_n(B,A)$ has rank n.*

$$C_T(B,A) = \begin{bmatrix} B & AB & \dots & A^{T-1}B \end{bmatrix}.$$

If the system is controllable, then for any initial state it is possible to design a unique set of shocks that will lead to a desired trajectory of states x_t.

Theorem 3 (System minimal realisation)**.** *The system $\{A,B,C,D\}$ is minimal if it is observable and controllable.*

Proof. See Kalman (1962) □

De Schutter (2000, p.332) describes the problem of minimal state space realisation (of a state-space model) as: "... given some data about linear time invariant system, find a state space description of minimal size, that describes the given data.”

The following theorem states how the minimal realisation problem is related to the initial condition identification.

Theorem 4. *If the order of the state-space system is minimal then we can uniquely recover the structural shocks $\{u_t\}_{t=1}^{T}$ and state variables $\{x_t\}_{t=0}^{T}$.*

Proof. The problem can be broken up in to two parts. First, if I know $\{y_t\}_{t=1}^{T}$ can I get a unique x_0, that is a unique $\{x_t\}_{t=0}^{T}$ that leads to x_0? This is equivalent to checking the observability condition. Second, knowing x_0 (and $\{x_t\}_{t=1}^{T}$), can I get a unique sequence of exogenous shocks $\{u_t\}_{t=1}^{T}$ that explains such trajectory? This is equivalent to checking the controllability condition.

Looking at the first problem, we solve the following system of equations:

$$
\begin{aligned}
y_1 &= Cx_0 + Du_1 \\
y_2 &= CAx_0 + CBu_1 + Du_2 \\
y_3 &= CA^2x_0 + CABu_1 + CBu_2 + Du_3 \\
&\vdots \\
y_T &= CA^{T-1}x_0 + CA^{T-2}Bu_1 + ... + CBu_{T-1} + Du_T
\end{aligned}
$$

In a matrix form

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix} - \begin{bmatrix} D & 0 & 0 & 0 & \dots & 0 \\ CB & D & 0 & 0 & \dots & 0 \\ CAB & CB & D & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ CA^{T-2}B & CA^{T-3}B & CA^{T-4}B & \dots & CB & D \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_T \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix} x_0 \tag{9}
$$

If the matrix on the right-hand side of the equation (the observability matrix) is left invertible, i.e. it has full column rank, then the system can be uniquely solved for x_0.

Looking at the second problem, we know x_0 and solve the system for the unique realisation of $\{u_t\}_{t=1}^T$:

$$\begin{aligned}
x_1 &= Ax_0 + Bu_1 \\
x_2 &= A^2x_0 + ABu_1 + Bu_2 \\
x_3 &= A^3x_0 + A^2Bu_1 + ABu_2 + Bu_3 \\
&\vdots \\
x_T &= A^Tx_0 + A^{T-1}Bu_1 + ... + ABu_{T-1} + Bu_T
\end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_T \end{bmatrix} - \begin{bmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^T \end{bmatrix} x_0 = \begin{bmatrix} B & 0 & 0 & \dots & 0 \\ AB & B & 0 & \dots & 0 \\ A^2B & AB & B & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{T-1}B & A^{T-2}B & A^{T-3}B & \dots & B \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_T \end{bmatrix} \tag{10}$$

The solution is unique if the matrix on the right-hand side is invertible. It is invertible if the controllability matrix $C_T(B,A)$ has full column rank. Thus is the state-space model is minimal we get unique trajectory for both $\{x_t\}_{t=0}^T$ and $\{u_t\}_{t=1}^T$ $\square$

4 An application

Lets have a look at an example now. For demonstration purposes, I use a simplified version of the new Keynesian business cycle model:

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(\varphi+\nu)(1-\zeta\beta)(1-\zeta)}{\zeta}\xi_t + u_{S,t} \tag{11}$$

$$\xi_t = E_t\xi_{t+1} + \frac{1}{\varphi}[r_t - E_t(\pi_{t+1})] + u_{D,t} \tag{12}$$

$$r_t = \phi_r r_{t-1} + (1-\phi_r)(\phi_\pi \pi_t + \phi_y \xi_t) + u_{r,t} \tag{13}$$

The Phillips curve (11) is firms' linearized pricing rule, where π_t is the aggregate price level inflation rate. The IS curve (12) is households' linearized Euler equation capturing the output ξ_t. The nominal side of the economy is controlled by the central bank's interest rate rule (13), where r_t is the nominal interest rate set at the period t. $u_{S,t}$, $u_{D,t}$, and $u_{r,t}$ are the supply (cost-push) shock, deman shock, and monetary-policy shock, respectively. All shocks are iid $N(0,\sigma^2_{u_i})$ for all $i = \{S,D,r\}$. The model's deep structural parameters (earlier denoted as θs) are $0 < \beta < 1$, $\nu > 0$, $\varphi > 0$, $\zeta > 0$, $0 \leq \phi_r < 1$, and ϕ_π and ϕ_ξ are such that the Taylor principle holds.

We can immediately see that ν and ζ cannot be identified. But as discussed in section 2, for our purposes we do not need to worry about it. The deep structural parameters itself are not that important. It is their product $\frac{(\varphi+\nu)(1-\zeta\beta)(1-\zeta)}{\zeta}$ that detemines the impulse response function.

The semi-structural form After solving the model for rational expectations, we receive the minimum-state variable (MSV) model, which represents the law of motion for ξ_t, π_t, and r_t. As discussed above, the MSV representation is important for recovering matrices A_0 and A_1, which have an essential place in the identification methodology. The MSV form for (11)-(13) is[4]

$$A_0 y_t = A_1 y_{t-1} + u_t,$$

where $y_t = \begin{bmatrix} \pi_t & \xi_t & r_t \end{bmatrix}'$, $u_t = \begin{bmatrix} u_{D,t} & u_{S,t} & u_{r,t} \end{bmatrix}'$, and

$$A_0' = \begin{pmatrix} a_{0,11} & 0 & a_{0,13} \\ a_{0,21} & a_{0,22} & a_{0,23} \\ a_{0,31} & a_{0,32} & a_{0,33} \end{pmatrix}, \quad A_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{1,33} \end{pmatrix}.$$

For model dynamics, it is essential that the semi-structural parameters $a_{0,ij}$ and $a_{1,ij}$ for all $i,j = 1,2,3$ can be uniquely estimated.

Case 1: $n = k = r$

To demonstrate the use of the identification methodology, it is the best to start with the simplest possible case. Lets begin with the special case where we observe all model variables, and the number of shocks is equal to the number of observables, i.e. $n = k = r = 3$. It is the simplest case, because all variables are observable and

[4] The full derivation is in Appendix A.

C is an identity matrix, The MSV solution is then also a SVAR(1) model, which allows for a direct application of Rubio-Ramirez et al.'s (2009) theory.

First, we form the transformation $f(A'_0, A'_1)$ by stacking A'_0 and A'_1

$$f(A'_0, A'_1) = \begin{bmatrix} A'_0 \\ A'_1 \end{bmatrix}.$$

Second, we re-order the equations, which are now captured by collums in $f(A'_0, A'_1)$, by putting the equation with most exclusionary restrictions as the first one. Here, it is the IS curve. Swapping the second column of $f(.)$ with the first one, we get

$$f(A'_0, A'_1) = \left[\begin{array}{ccc} 0 & a_{0,11} & a_{0,13} \\ a_{0,22} & a_{0,21} & a_{0,23} \\ a_{0,32} & a_{0,31} & a_{0,33} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{1,33} \end{array}\right]$$

Third, we represent the zero restrictions in $f(A'_0, A'_1)$ by Q_j matrices that form the nulspace with $f(A'_0, A'_1)$. Each Q_j captures the exclusionary restrictions in the j's column of $f(A'_0, A'_1)$. For the first column, the IS curve, we have Q_1

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the second and third column, the Phillips curve and the policy rule, respectivelly, we form

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The fourth and final step is to construct the matrices $M_j(f(A_0', A_1'))$ from Theorem 5 for all $j = 1,2,3$. Skipping the zero rows, we get

$$M_1 = \left(\begin{array}{ccc} 0 & a_{0,12} & a_{0,13} \\ 0 & 0 & a_{1,33} \\ \hline 1 & 0 & 0 \end{array} \right), M_2 = \left(\begin{array}{ccc} 0 & 0 & a_{1,33} \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), M_3 = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

Again, each M_j represents individual model equation, which are ordered as the columns in $f(.)$. The rank of M_j can be interpreted as the check of partial identification of an individual equation. If $rank(M_j) = n$, one can say that the shock associated with the j-equation is identified. Clearly, $rank(M_j) = 3$ for all js, and thus we can conclude that the semi-structural model *is identified*. Note that the identification comes from the lagged interest rate, r_{t-1} in the policy rule. If $\phi_r = 0$ then the model does not produce enough instruments to identify the Phillips curve, the second equation, and the rank condition would be violated, $rank(M_2) = 2$.

In the following two cases, I demonstrate the usefulness of the methodology in the case when modellers have available less observable variables than there are model variables. It is interesting, because then the inverted DSGE model will have an infinite SVAR representation. I show that the location of shocks matters for identification in such a case.

Case 2: $n = k < r$

Lets reduce the number of shocks and observable variables.[5] The choice of shocks and model observable variables is purposefull. I abstract from any economic meaning the choice may make. I only wish to show that the location of shocks matters for their identifiability.

No demand shock: $\sigma_{u_D}^2 = 0$

I will assume now that I do not observe the output ξ (or output gap), and I will treat the IS curve as deterministic, and put $\sigma_{u_D}^2 = 0$. As a model designer, I make the

[5] This exercise is equivalent to compounding the shocks, so that their number is reduced to n in the measurement equation (6).

choice that the model economy will only be driven by supply (cost-push) shocks, and monetary policy shocks. Because we have $n = k = 2$ and $r = 3$, the inverted model will be of an infinite order. Now we have to construct $f(A_0, A_1, A_2)$ as in section 3. In this example, A_2 turns out to be a zero matrix, so I am going to ommit it.

Following the four-step procedure above, the $f(A_0', A_1')$ representation is

$$f(A_0', A_1') = \left(\begin{array}{cc} a_{0,11} & a_{0,21} \\ a_{0,12} & a_{0,22} \\ \hline 0 & 0 \\ a_{1,12} & a_{1,22} \end{array} \right).$$

Corresponding $M_j(f(A_0, A_1))$ for $j = 1, 2$ are

$$M_1 = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \end{array} \right), \quad M_2 = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array} \right).$$

Because $rank(M_1) = 1$ and $rank(M_2) = 2$, the semi-structural model without a demand shock and observable output *cannot be identified*. The Phillips curve and the cost-push shock are lacking identification. What happens if we change the assumption about shocks?

No monetary policy shock: $\sigma_{u_R}^2 = 0$

Lets consider exactly the same set-up as in the previous case, but now the theoretical model includes demand shocks, $\sigma_{u_D}^2 > 0$, but there are not any monetary policy shocks, $\sigma_{u_R}^2 = 0$, and the policy rule holds exactelly. We still observe only inflation and interest rates. Again we have

$$f(A_0', A_1') = \begin{pmatrix} a_{0,11} & a_{0,21} \\ a_{0,12} & a_{0,22} \\ 0 & 0 \\ 0 & a_{1,22} \end{pmatrix},$$

and correspondingly $M_j(f(A_0,A_1))$ for $j=1,2$, which are

$$M_1 = \left(\begin{array}{cc} 0 & a_{1,22} \\ \hline 1 & 0 \end{array}\right), \quad M_2 = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right).$$

We immediately see that $rank(M_1) = rank(M_2) = 2$, and can conclude that the semi-structural model *is identified*. The impulse response response of observable variables to structural shocks is uniquelly informed by the data. Thus changing the assumption about the shocks results in impulse response identifiability.

5 Conclusion

I showed how to applied the SVAR identification methodology by Rubio- Ramirez, Waggoner and Zha (2007) to DSGE models. I used the methodology to determine if the model's semi-structural form is globally identifiable, and if we are able to recover data consistent and unique impulse responses. If there is no other observationally equivalent set of structural shocks that would explain the data, such model is said to have unique (identified) impulse responses. The methodology consists of a few matrix operations and evaluations, and is straightforward to code and apply. Because no evaluation of likelihood function is involved, the methodology is computationally cheap. The methodology is very powerful, and provides useful piece of information for DSGE model developers. As they are many kinds of structural shocks that would classify as demand, supply or policy shocks, their identifiability may serve as one of the criteria for a discrimination among them.

6 References

Canova F., and L. Sala (2009), "Back to Square One: Identification Issues in DSGE Models," *Journal of Monetary Economics* Vol. 56, pp.431-449

Fernández-Villaverde J., J.F. Rubio-Ramírez, T.J. Sargent and M.W. Watson (2007),"ABCs (and Ds) of Understanding VARs," *The American Economic Review* Vol. 97, No. 3

Fukač M., A. Pagan, and V. Pavlov (2006), "Econometric Issues Arising from DSGE Models," paper presented to the Macroeconomics and Model Uncertainty Conference, Wellington. Available at
http://www.rbnz.govt.nz/research/workshops/27jun06/2647404.pdf

Kalman, R.E. (1963), "Mathematical Description of Linear Dynamical Systems," *SIAM Journal of Control*, Vol. 1, No. 2, pp. 152-192

Ljung L. (1999), "System Identification: Theory for the Users." Second Edition.

Prentice Hall, New Jersey.

Preston A.J. (1978), “Concepts of Structure and Model Identifiability for Econometric Systems," in: A.R. Bergstrom et al., *Stability and Inflation* (Wiley, 1978), p.275-297.

Rubio-Ramirez J.F., D.F. Waggoner, and T. Zha (2007), “Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference," mimeo

Rothenberg T. (1971), “Identification in Parametric Models," *Econometrica* Vol. 16, pp.577-591.

De Schutter, B. (2000), “Minimal State-Space Realization in Linear System Theory: An Overview,” *Journal of Computational and Applied Mathematics*, Special Issue on Numerical Analysis in the 20th Century - Vol. I: Approximation Theory, Vol. 121, No. 1-2, p. 331-354

Appendices

A Proof of Theorem 1

Proof. If $A_0 = \tilde{A}_0 P$ and $A_+ = \tilde{A}_+ P$, then

$$B = A_+ A_0^+ = \tilde{A}_0 P P^{-1} \tilde{A}_0^+ = \tilde{A}_+ \tilde{A}_0^+ = \tilde{B}$$

$$\Sigma = (A_0 A_0')^+ = (\tilde{A}_0 P P' \tilde{A}_0')^+ = (\tilde{A}_0 \tilde{A}_0')^+ = \tilde{\Sigma}$$

If they are observationally equivalent then $A_+ A_0^+ = \tilde{A}_+ \tilde{A}_0^+$ and $(A_0 A_0')^+ = (\tilde{A}_0 \tilde{A}_0')^+$. From the latter it follows that

$$\begin{aligned}
(A_0 A_0')^+ &= (\tilde{A}_0 \tilde{A}_0')^+ \\
A_0^{+\prime} A_0^+ &= \tilde{A}_0^{+\prime} \tilde{A}_0^+ \\
A_0' A_0^{+\prime} A_0^+ &= A_0' \tilde{A}_0^{+\prime} \tilde{A}_0^+ \\
A_0^+ &= A_0' \tilde{A}_0^{+\prime} \tilde{A}_0^+ \\
A_0' (A_0 A_0')^{-1} &= (\tilde{A}_0^+ A_0)' \tilde{A}_0^+ \\
A_0' &= (\tilde{A}_0^+ A_0)' (\tilde{A}_0^+ A_0) A_0' \\
(A_0 A_0')' [(A_0 A_0')']^{-1} &= (\tilde{A}_0^+ A_0)' (\tilde{A}_0^+ A_0)(A_0 A_0')' [(A_0 A_0')']^{-1} \\
I &= (\tilde{A}_0^+ A_0)' (\tilde{A}_0^+ A_0)
\end{aligned}$$

Therefore $P = \tilde{A}_0^+ A_0$ is orthogonal and $\tilde{A}_0 P = A_0$; that is

$$\begin{aligned}
P &= \tilde{A}_0^+ A_0 \\
\tilde{A}_0 P &= \tilde{A}_0 \tilde{A}_0^+ A_0 \\
\tilde{A}_0 P &= \tilde{A}_0 \tilde{A}_0' (\tilde{A}_0 \tilde{A}_0')^{-1} A_0 \\
\tilde{A}_0 P &= A_0
\end{aligned}$$

Using this result for $A_+A_0^+ = \tilde{A}_+\tilde{A}_0^+$, we obtain

$$\begin{aligned}
A_+A_0^+ &= \tilde{A}_+\tilde{A}_0^+ \\
A_+A_0'(A_0A_0')^{-1} &= \tilde{A}_+\tilde{A}_0^+ \\
A_+A_0' &= \tilde{A}_+\tilde{A}_0^+A_0A_0' \\
A_+(A_0A_0')' &= \tilde{A}_+\tilde{A}_0^+A_0(A_0A_0')' \\
A_+(A_0A_0')'[(A_0A_0')']^{-1} &= \tilde{A}_+\tilde{A}_0^+A_0(A_0A_0')'[(A_0A_0')']^{-1} \\
A_+ &= \tilde{A}_+\tilde{A}_0^+A_0 \\
A_+ &= \tilde{A}_+P
\end{aligned}$$

□

B Proof of Theorem 2

Proof. With a minor modification, the proof is the same as in RWZ (2007, p.15). Let $q_j = Pe_j - p_{jj}e_j$, where $P = \tilde{A}_0^+A_0$ is $(k \times k)$ orthogonal matrix, p_j is the first column of P with non-zero off-diagonal elements, e_j is the j^{th} column of an identity matrix I_k. To proof the theorem, it is sufficient to show, that the rank of $M_j(f(A_0,A_0))$ is strictly less than k. Since $q_j \neq 0$, it suffices to show that $M_j(f(A_0,A_+))q_j = 0$. Because both (A_0,A_+) and (A_0P,A_+P) are in R, by construction of Q_j it holds that $Q_jf(A_0,A_+)q_j = 0$. Thus the upper block of $M_j(f(A_0,A_+))$ is zero. The lower block $[I \quad 0]q_j$ is also equal to zero, because I is $(j \times j)$, and first j elements of e_j are zero. □

C Proof of Lemma 1

TBA

D Solution to examples in Section 3

Case 1: $r = n = k$

$$x_t' B_0 = x_{t-1} C + (E_t x_{t+1})' D + u_t',$$

$x_t = [\pi_t \quad \xi_t \quad r_t]$, B_0, C, and D are 3×3 matrices of the semi-structural form parameters, and u_t is iid $N(0,1)$.

The MSV solution to the model is

$$x_t' A_0 = x_{t-1} A_1 + u_t',$$

where

$$\begin{aligned} B_0 - C A_0^{-1} D &= A_0 \\ A_1 &= C \end{aligned}$$

Now, the task is to characterize the structure of A_0. We know that
$B_0 = \begin{pmatrix} + & 0 & + \\ + & + & + \\ 0 & + & + \end{pmatrix}, D = \begin{pmatrix} + & + & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}.$

Let $A_0^{-1} = \begin{pmatrix} a^*_{0,11} & a^*_{0,12} & a^*_{0,13} \\ a^*_{0,21} & a^*_{0,22} & a^*_{0,23} \\ a^*_{0,31} & a^*_{0,32} & a^*_{0,33} \end{pmatrix}$, and substitute to (3) to get an idea how the structure of A_0 looks like and we can apply the counting rule from Tao's paper. We get

$$\begin{pmatrix} + & 0 & + \\ + & + & + \\ 0 & + & + \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ + & + & + \end{pmatrix} \begin{pmatrix} + & + & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{pmatrix} = A_0,$$

and from there $A_0 = \begin{pmatrix} + & 0 & + \\ + & + & + \\ + & + & + \end{pmatrix}$.

Now we can apply Theorem 2 to check the general rank condition.

$$f(A_0,A_1) = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} + & 0 & + \\ + & + & + \\ + & + & + \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix} = \begin{pmatrix} 0 & + & + \\ + & + & + \\ + & + & + \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now I have to check the rank condition:

$$M_1 \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} & Q_1 \left(\begin{matrix} A_0 \\ A_1 \end{matrix} \right) & \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & + & + \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_2 \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} & Q_1 \left(\begin{matrix} A_0 \\ A_1 \end{matrix} \right) & \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We can see that $rank(M_1) = 3$ and $rank(M_2) = 3$, and thus we can conclude that the model is globally identified.

Case 2: $n = k < r$

I.) No demand shock: $\sigma^2_{u_D} = 0$

In this exercise, I assume that the output gap ξ is unobservable, and there is no demand shock $u_{D,t}$. The solved structural model than takes the form

$$A_0 x_t = A_1 x_{t-1} + F u_t.$$

$$A_0 = \begin{pmatrix} + & 0 & + \\ + & + & + \\ + & + & + \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} y_t &= Cx_t \\ &= CA_0^{-1}A_1 x_{t-1} + CA_0^{-1}Fu_t \\ &= \mathbf{C}x_{t-1} + \mathbf{D}u_t \end{aligned}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 & + \\ 0 & 0 & + \end{pmatrix}, \mathbf{D} = \begin{pmatrix} + & + \\ + & + \end{pmatrix}.$$

Solving for a SVAR representation of the DSGE model we get

$$A_0 y_t = A_1 y_{t-1} + u_t$$

$$A_0 = \mathbf{D}^+ = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, A_1 = \mathbf{D}^+\mathbf{CBD}^+ = \begin{pmatrix} 0 & + \\ 0 & + \end{pmatrix}.$$

$$f(A_0', A_1') = \begin{pmatrix} + & + \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} Q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$M_1 = \left(\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \end{array}\right), M_2 = \left(\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right).$$

We can see that $rank(M_1) = 1$ and $rank(M_2) = 2$ – the model is *not* identified.

II.) No monetary policy shock: $\sigma_{u_R}^2 = 0$

I assume the same setting as in Case 2.I, with the only difference, that there is no monetary policy shock in the Taylor rule $u_{I,t}$, but there is a demand shock, $u_{D,t}$.

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 & + \\ 0 & 0 & + \end{pmatrix}, \mathbf{B} = \begin{pmatrix} + & + \\ + & + \\ + & + \end{pmatrix}.$$

Similarly as above, $A_0 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 0 & + \end{pmatrix}$.

$$f(A_0', A_1') = \begin{pmatrix} + & + \\ + & + \\ 0 & 0 \\ 0 & + \end{pmatrix}, Q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & + \\ \hline 1 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{pmatrix}$. $rank(M_1) = 2$ whereas $rank(M_2) = 2$ – the model *is* identified.

E Matrix pseudoinverse

Definition 5 (Matrix pseudoinverse)**.** *For a matrix A whose elements are real numbers, its pseudoinverse A^+ is a unique transformation which meets the following criteria:*

$$\begin{aligned} AA^+A &= A; \\ A^+AA^+ &= A^+; \\ (AA^+)' &= AA^+; \\ (A^+A)' &= A^+A; \end{aligned}$$

Some useful properties:

- Pseudoinversion is reversible: $(A^+)^+ = A$;
- $(A')^+ = (A^+)'$;
- $A^+ = A^+A^{+\prime}A'$;
- $A^+ = A'A^{+\prime}A^+$;
- If A is of full column rank then $A^+ = (A'A)^+A'$, and $A^+A = I$; A^+ is left inverse of A;
- If A is of full row rank then $A^+ = A'(AA')^+$, and $AA^+ = I$; A^+ is right inverse of A;
- If A is square, non-singular matrix then $A^+ = A^{-1}$.