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Regulated Revenue Sharing Rules as Compensation for Essential Inputs

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Abstract

Essential inputs are an important topic of debate for economics. One common essential input is intellectual property, in the form of either patents or copyrights, which the producers of goods and services for final consumption must necessarily purchase from the input supplier. The ensuing monopoly power of the input supplier leads in many cases to controversial outcomes, in which social inefficiencies can occur. In much of the literature on the economics of intellectual property, it is assumed that the right holder is remunerated either by a fixed payment or by a payment that amounts to an additional marginal cost to the user, or both. However, in some significant instances in the real-world, right holders are constrained by regulators to use (or may choose to use) a compensation scheme that involves revenue sharing. That is, the right holder takes as remuneration a part of the user's revenue. In essence, the remuneration is set as a tax on the user's revenue. This paper analyses such remuneration mechanisms, establishing and analysing the optimal tax rate, and also the Nash equilibrium tax rate that would emerge from a fair and unconstrained bargaining problem. The second option provids a rate that may be useful for regulatory authorities.

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1 Introduction

In much of the literature on contracts for the use of intellectual property as an essential input to a production process, the royalty compensation is set as either a fixed cost to the user or as an element of additional marginal cost (see, for example, Kamien and Tauman (1986), Wang (1998), Fosfuri and Roca (2004), and Sen (2005)). This is reasonable for some, but not all settings. While the royalty earnings of artists, singers and literary authors are often set as above, in other cases the royalty payment that is allocated to the right holder as compensation is set as a tax upon the user´s revenue. This, for example, is the typically the case for music that is played publicly on the radio. The literature on revenue sharing contracts is quite extense, although it concentrates almost exclusively upon correcting for incentive effects along the supply chain (see, for example, Cachon and Lariviere (2005), and Wang et al. (2004)). There is also a small literature that examines revenue sharing in the video rental market (Dana and Spier (2001), Mortimer (2008)).

Attempts to examine theoretically the optimal royalty tax for the right holder to set are rare. Michel (2006) considers the case of a copyright holder supplying music to a record label in exchange for a share in revenue. While the optimal revenue share is not discussed, Michel does consider the Nash bargaining revenue share. However, the main thrust of the Michel paper is to consider the comparative statics of copyright piracy and Internet file sharing upon the bargained contract. Although Michel is able to study certain comparative static features of the Nash bargaining solution, the complexity of the environment assumed by Michel does not allow a closed form solution to the problem to be found. Second, Marchese and Ramello (2011), in the context of religious messages, study a game in which a copyright holder (the Church) supplies an essential input to a distributor while simultaneously competing in the output market. However, the objective of the right holder in that paper is not to maximise income, but rather to maximise diffusion.

In the present paper I put forward an analysis of the remuneration that the supplier of an essential input when the payment to the supplier is restricted to being a revenue tax. The supplier in the present paper does act with the objective of maximising income, and so the type of scenario here is likely to be relevant to a large number of real-world settings in which essential inputs are involved. Above all, aside from considering the unrestricted monopoly price that the supplier would optimally charge, the paper looks closely at the Nash bargining outcome, and attempts to provide useful solutions to the dilemma faced by regulators of such transactions – how can a fair, and also relatively non-complex, price be determined?

2 Modelling assumptions

We shall assume that the setting is one of a single supplier of an essential input[1] and a single user of that input. The input that is supplied is essential to the user's production process, in the sense that without access to the input, the user is unable to produce any output for consumers. We assume that the user acts as a monopolist in the market for the final good that he produces, which we denote by x, and likewise the input supplier acts as a monopolist in the supply of the essential input. For reasons that are determined exogenously (e.g. the prevailing legal environment), the remuneration of the supplier is set as a tax on the revenue that is generated by the user.[2] We assume that the demand for the final consumption good produced by the user is linear, that is the price at which the good is sold is given by

$$p(x) = 1 - bx$$

and that the user's production process is also linear (i.e. it is characterised by constant marginal cost), with marginal cost equal to c, where at all times $c < 1$. For the initial analysis, the user is assumed to face no fixed costs, and his objective is to maximise monetary profits. The issue of fixed costs will be discussed below in a separate section. Note that the demand vertical intercept has been set to 1. This is without loss of generality, and simply sets the units of measurement for the problem.

We are only interested in the case in which the essential input is already in existence, rather than the case in which units of it are produced as required. This is, for example, the case of

[1] The term "input supplier" is used liberally. More precisely, the supplier of the essential input could be a collecting society that represents a whole group of copyright holders, and the essential input could be an entire repertory of individual elements.

[2] Actually, it can be shown that a revenue tax will always dominate a per-unit royalty (see Marchese and Ramello, 2011). It is also true that for each feasible revenue sharing rule, there is an equivalent profit sharing rule. Revenue sharing as opposed to profit sharing avoids the complications of asymmetric information regarding costs.

intellectual property. For that reason, the input supplier is assumed to face no costs at all in the supply of the input to the user, and so her objective is to maximise income.[3]

3 Analysis

The compensation for the input supplier is a revenue tax,[4] and so for any given choice of output x, assuming that the tax is set at t, the user ends up with a profit of

$$\begin{aligned} \pi(t) &= (1-t)(1-bx)x - cx \\ &= \left[(1-t)(1-bx) - c\right] x \end{aligned} \tag{1}$$

The input supplier would get an income of

$$R(t) = t(1-bx)x \tag{2}$$

Once the tax rate has been set, the user will then freely choose his level of output, x, to maximise his profits. Maximising $\pi(t)$ with respect to x gives

$$x^*(t) = \frac{(1-t)-c}{2b(1-t)} \tag{3}$$

The optimal level of output is a decreasing and concave function of the tax rate:

$$x^{*\prime}(t) = -\frac{c}{2b(1-t)^2} < 0 \qquad x^{*\prime\prime}(t) = -\frac{c}{b(1-t)^3} < 0 \tag{4}$$

Our principal analysis is of the input supplier's income function. We are interested in considering the value of t that will maximise it, subject to the user's profit remaining non-negative. It is useful to separate the analysis of $t = 1$ from the analysis of $t < 1$. Clearly, from (3), if $t = 1$, the optimal level of output is not defined. However, there is a correspondence between $c = 0$ and $t = 1$. The reason for this is the following. Assume that $c = 0$. Then, if $t < 1$, the optimal level of output would be $\frac{(1-t)}{2b(1-t)} = \frac{1}{2b}$. What about if $t = 1$? To find out what the optimal output would be we need to analyse the optimal level of output in limit as t approaches 1. But using L'Hôpital's

[3] Both the input supplier and the input user are assumed to be risk neutral. In fact, the environment is assumed to be risk free. This is done in order to simplify the analysis as much as possible, given the objective of finding a simple, workable, and useful sharing rule.

[4] Of course this is exactly the same as a tax on price (an ad valorem tax), at least in the context of the present paper. The equivalence is due to the assumption of a demand curve, and a single price for all customers.

theorem, it turns out that

$$\lim_{t\to 1}\frac{(1-t)}{2b(1-t)} = \frac{-1}{-2b} = \frac{1}{2b}$$

Thus, independently of what level of t is used, when $c = 0$ the user will set output at $\frac{1}{2b}$. But this implies that the total amount of revenue earned is a constant, equal to $\left(1 - b\frac{1}{2b}\right)\left(\frac{1}{2b}\right) = \frac{1}{4b}$. Since the user does not alter output (or revenue) in response to a change in the tax rate, the best choice of the input supplier is clearly to set the tax rate as high as possible in this case, and so we get $t^*(c)|_{c=0} = 1$. With this tax rate, the user produces a strictly positive level of output, and generates a strictly positive level of revenues, which the input supplier then takes entirely as the royalty payment. The user ends up with 0, and thus is indifferent between producing optimally and not producing at all. Given this, we shall only be concerned from now on with scenarios of $c > 0$. Since it is also necessary that $c < 1$, in all that follows we will only be concerned with $0 < c < 1$.

The non-negativity assumption on the user's profit implies that there is a maximum tax rate that the input supplier cannot exceed. To calculate this maximum, consider again the user's profit for any tax rate t, assuming of course that the user sets his output choice optimally given that tax rate;

$$\pi(t) = \left[(1-t)(1-bx^*(t)) - c\right] x^*(t)$$

The effect of a change in t is given by the first derivative of $\pi(t)$, which is

$$\pi'(t) = \left[(1-t)(1-bx^*(t)) - c\right] x^{*\prime}(t) - (1-bx^*(t))x^*(t) - (1-t)bx^{*\prime}(t)^2$$

Since we must have $\pi(t) > 0$, it happens that $[(1-t)(1-bx^*(t)) - c] > 0$, and since $x^{*\prime}(t) < 0$ the first term is negative. The second term is also negative since $(1-bx^*(t)) > (1-t)(1-bx^*(t)) - c > 0$, and clearly the third term is also negative. Thus for any t, once output is set optimally, $\pi'(t) < 0$.

Theorem 1 *The maximum tax rate that the input supplier can set is $\bar{t} = 1 - c$.*

Proof. The fact that the user's profit is decreasing in t implies that the maximum tax rate that can be set, $\bar{t}$, satisfies

$$\pi(\bar{t}) = \left[(1-\bar{t})(1-bx^*(\bar{t})) - c\right] x^*(\bar{t}) = 0$$

So either $(1-t)(1-bx^*(t))-c=0$, or $x^*(t)=0$. The tax rate that would set optimal output to 0 is easily calculated from (3) as

$$\bar{t}_1 = 1 - c$$

while the tax rate that sets $(1-\bar{t}_2)(1-bx^*(\bar{t}_2))-c=0$ is[5]

$$1 - \bar{t}_2 = c$$

But this is the same tax rate that sets optimal output to 0, i.e. $\bar{t}_2 = \bar{t}_1$. Thus the limit tax rate for the input supplier is $\bar{t} = 1 - c$. ■

While it is necessary to go throught the above analysis of the maximum feasible tax rate, it is also worthwhile to note the following:

Theorem 2 *Assuming $c > 0$, the input supplier will never want to set the tax rate at (or of course above) the maximum feasible level.*

Proof. We are always assuming $c < 1$, which implies that it is always possible for the activities of the user to generate positive profits, and of course positive revenue. However, if the tax rate on revenue is set at the maximum feasible level identified in Theorem 1, the optimal response of the user is to set output at 0, and thus total revenue generated is also 0. This means that the input supplier will earn no royalty income at all. However, setting the tax rate at a smaller level, such that now the user does produce a positive level of output, and generates a positive total revenue, will imply a positive level of earnings for the input supplier. Thus, it will never be optimal for the input supplier to set the tax rate at the maximum level. ■

Theorem 2 has two implictions. First, since we are only concerned with scenarios of $c > 0$, we know that we always have $t^*(c) < 1 - c < 1$. This is important as it implies that we can always safely assume $1 - t > 0$, so that divisions by $1 - t$ (which will frequently be done) are valid. Second, we can always safely ignore the restriction that the optimal tax rate set by the input supplier should satisfy $t^* \leq 1 - c$, and then go ahead with studying the unconstrained optimisation problem for the input supplier, which is

$$\max_t R(t) = t(1 - bx^*(t))x^*(t)$$

[5] See section 1 of the appendix for all of the steps used in finding this equation.

Substituting (3) into (2), we can then see that the input supplier's objective function is[6]

$$\begin{aligned} R(t) &= t(1-bx^*(t))x^*(t) \\ &= \frac{t((1-t)^2-c^2)}{4b(1-t)^2} \end{aligned} \tag{5}$$

We would like to maximise (5) with an appropriate choice of t. In order to do that, we firstly need to investigate the shape of this function. The first derivative of $R(t)$ is[7]

$$R'(t) = \left(\frac{1}{4b}\right)\left(1 - \frac{(1+t)c^2}{(1-t)^3}\right) \tag{6}$$

The second derivative is

$$R''(t) = -\left(\frac{c^2}{4b}\right)\left(\frac{t+2}{(1-t)^4}\right) < 0 \tag{7}$$

Since the second derivative is negative, $R(t)$ is concave in t, and so the unconstrained optimal t is found where the first derivative is 0. That is

$$t^* \leftarrow 1 = \frac{(1+t^*)c^2}{(1-t^*)^3} \tag{8}$$

We can analyse this equation in a variety of ways, but let us simply write $f(c,t) = 1 - \frac{(1+t)c^2}{(1-t)^3}$, so that we have $f(c,t^*) = 0$. The analysis of the optimal tax rate can then be done by studying the roots of $f(c,t)$.

Theorem 3 *If $c > 0$, there is a single root of $f(c,t)$, and that root occurs at a t that satisfies $0 < t < 1$.*

Proof. Note the following:

$$\begin{aligned} f(c,0) &= 1 - c^2 > 0 \\ \lim_{t\to 1} f(c,t) &= -\infty < 0 \\ \frac{\partial f(c,t)}{\partial t} &= -\frac{c^2(1-t)^3 + (1+t)c^2 3(1-t)^2}{(1-t)^6} \\ &= -2c^2\frac{(2+t)}{(1-t)^4} < 0 \end{aligned}$$

Thus, the function $f(c,t)$ starts out positive at $t = 0$ and it is negative as t approaches 1. So there is at least one root of $f(c,t)$ for a t between 0 and 1. However, since the slope of the function is everywhere negative, there can only be a single root. ■

[6] See section 2 of the appendix for the working.

[7] The working appears in section 3 of the appendix.

The shape of the function $f(c,t)$ is shown in Figure 1.[8] The root of the function, t^*, is the unique optimal tax rate for the problem. For example, the graph in Figure 1 corresponds to $c = \frac{1}{2}$, in which case we get $t^* = 0.3106$.[9]

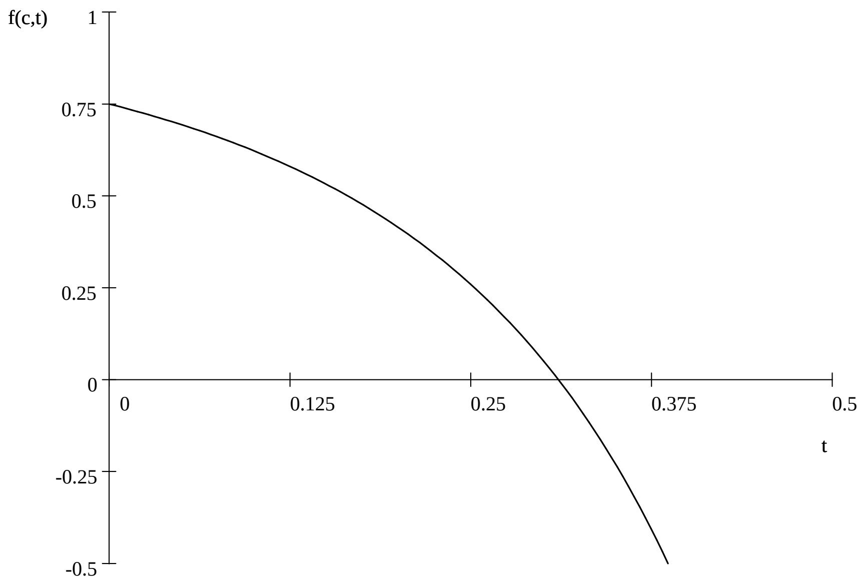

Figure 1: The shape of $f(c,t)$

Given that there is a single root of the function $f(c,t)$, we can find it algebraicly as the solution of (8). While not at all easy to do for a third order equation, this is certainly possible. Using the mathematical package DERIVE, the solution turns out to be

$$t^*(c) = 1 + \frac{\sqrt{3}c^{\frac{2}{3}}\left(\sqrt{(c^2+27)} - 3\sqrt{3}\right)^{\frac{1}{3}}}{3} - \frac{\sqrt{3}c^{\frac{2}{3}}\left(\sqrt{(c^2+27)} + 3\sqrt{3}\right)^{\frac{1}{3}}}{3}$$

The graph of $t^*(c)$, for all values of c between 0 and 1 is given in Figure 2. It is decreasing and convex on the range of feasible values of c.

[8] Throughout this paper, graphs are used to illustrate the equations derived. All of the graphs are computer generated from the actual equations in the text, and so they are completely accurate representations of each equation. The general characteristics of most of the equations we analyse can also be examined by recourse to the implicit function theorem. However, given that we have the possibility of accurate drawings of the graphs of the equations, which clearly indicate such aspects as slope and curvature, the mathematical analysis will be omitted.

[9] Note that for this example, the constraining maximum feasible tax rate is $1 - c = \frac{1}{2}$, the unconstrained optimum is indeed the global optimum for the problem.

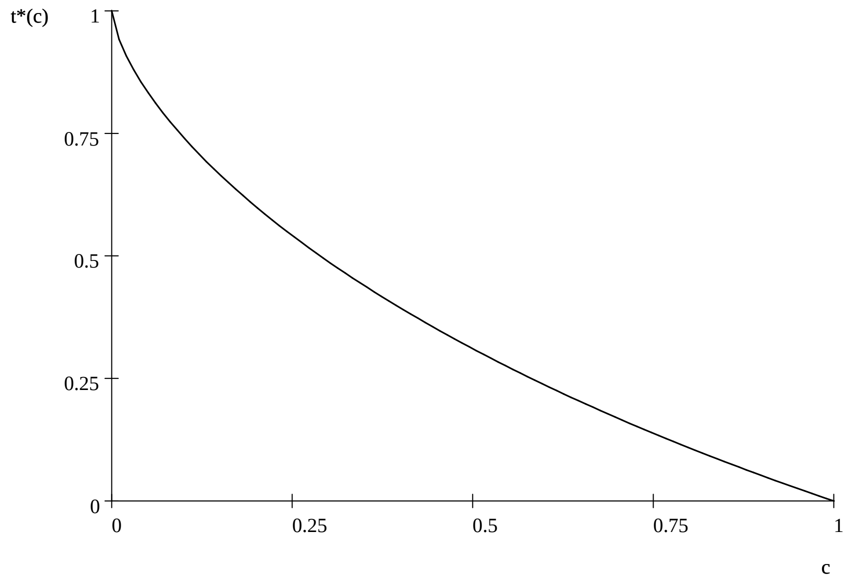

Figure 2: The optimal revenue tax, $t^*(c)$

4 Sharing of market surplus

In the above, we have simply performed an analysis of the optimal tax rate, from the perspective of the input supplier. We might also wonder about how the total market surplus is shared between the two parites under such a revenue sharing model.

To analyse surplus sharing, it is useful to express the user´s profit as a function of the royalty payment;[10]

$$\pi(t) = R(t)\left(\frac{1-t}{t}\right) - x^*(t)c \tag{9}$$

Given this, for any tax rate t, the ratio of user profit to input supplier income is

$$\begin{aligned}\frac{\pi(t)}{R(t)} &= \frac{R(t)\left(\frac{1-t}{t}\right) - x^*(t)c}{R(t)} \\ &= \left(\frac{1-t}{t}\right) - \frac{x^*(t)c}{R(t)}\end{aligned}$$

But since $R(t) = t(1-bx^*(t))x^*(t)$, we can write

$$\begin{aligned}\frac{\pi(t)}{R(t)} &= \left(\frac{1-t}{t}\right) - \frac{x^*(t)c}{t(1-bx^*(t))x^*(t)} \\ &= \left(\frac{1-t}{t}\right) - \frac{c}{t(1-bx^*(t))} \qquad (10)\end{aligned}$$

[10] The working for this is again given in section 4 of the appendix.

Substituting for the optimal output level and simplifying,[11] we get

$$\frac{\pi(t)}{R(t)} = \left(\frac{1-t}{t}\right)\left[\frac{(1-t)-c}{(1-t)+c}\right] \tag{11}$$

The graph of $\frac{\pi(t^*(c))}{R(t^*(c))}$ is displayed in Figure 3 as a function of c. It is a strictly increasing and concave graph. At $c = 1$, the height of the graph is 0.5.

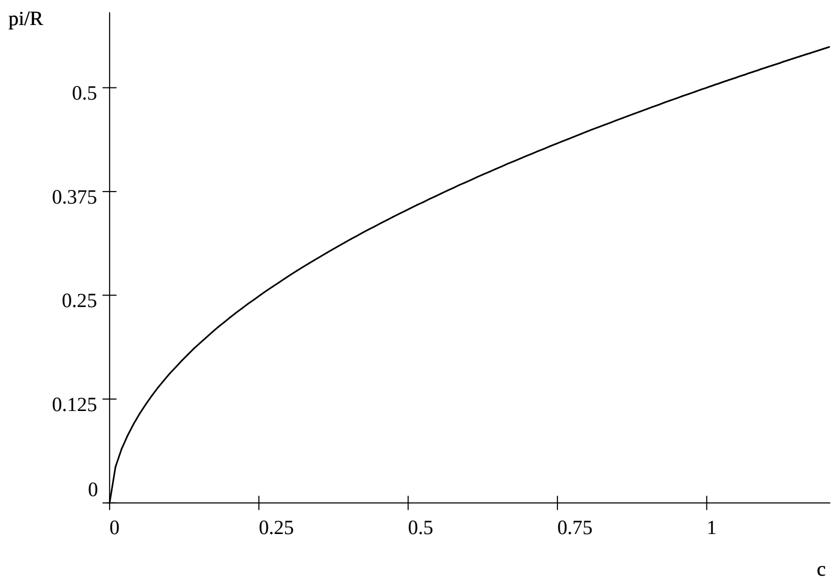

Figure 3: The shape of $\frac{\pi(t^*(c))}{R(t^*(c))}$

From Figure 3, we can establish the following result:

Theorem 4 *For all values of c, the input supplier earns at least twice as much as the user; $R(t^*) > 2\pi(t^*)$.*

Proof. The proof is evident from the fact that $\frac{\pi(t^*(c))}{R(t^*(c))}$ is located everywhere below the value $\frac{1}{2}$ over the relevant range of values of c. ■

5 The Nash bargaining solution

The model above has assumed that the input supplier can set the rate optimally. We noted that when this is done in any scenario in which positive profits can be made ($c < 1$), the input supplier will end up earning significantly more of the total surplus than will the user. In fact this may not

[11] See section 5 of the appendix for this working.

be unreasonable. Firstly, the input supplier is supplying an essential input to the user's production process, and as such it may be considered reasonable that the input supplier should earn most of the surplus that is created, since without her input, no surplus at all is created. Secondly, we are allowing the user to act as an unconstrained monopolist in the market for his output, and so it seems fair to allow the same consideration to the input supplier.

Never-the-less, if the monopoly power of the input supplier were thought to be giving her an unfair advantage, one could appeal to some other way of establishing the revenue sharing tariff rate. One logical option might be to appeal to the deal that would be struck in an unconstrained bargaining game with symmetric bargaining powers. In order to model such a game, it is habitual to use the Nash bargaining model (Nash (1950)), which seeks to find the t that maximises the Nash product, $N(t) = \pi(t)R(t)$.[12] In this section we look at this option.

In order to calculate the Nash product, it is useful to firstly re-write the user´s profit function. Starting from (1), we can substitute in the optimal output and simplify to get

$$\pi(t) = \left(\frac{1}{4b}\right)\left(\frac{((1-t)-c)^2}{(1-t)}\right) \tag{12}$$

Deriving this, we find that[13]

$$\pi'(t) = -\frac{R(t)}{t}$$

Thus, the derivative of the Nash product is

$$\begin{aligned} N'(t) &= \pi'(t)R(t) + \pi(t)R'(t) \\ &= -\frac{R(t)^2}{t} + \pi(t)R'(t) \end{aligned} \tag{13}$$

Notice that, if the Nash product is concave, then its maximum occurs at t^N such that[14]

$$\frac{R(t^N)^2}{t^N} = \pi(t^N)R'(t^N) > 0$$

[12] Of course the Nash bargaining model would really maximise the product of the difference between each utility and the corresponding disagreement utility. But for the case at hand, disagreement implies no deal at all, in which case both parties earn 0.

[13] The working is in section 6 of the appendix.

[14] It is interesting to note that, from this first-order condition, it can be established that $\frac{\pi(t^N)}{R(t^N)} = \frac{R(t^N)}{t^N R'(t^N)} = \frac{1}{\varepsilon}$, where ε is the elasticity of the input supplier's revenue to the tax rate.

This implies that it must be that $R'(t^N) > 0$, and since $R(t)$ is concave in t, it holds that $t^N < t^*$. So, assuming that the Nash product has a maximum at a concave point, the tax rate implied is less than the optimal tax rate for the copyright holder. That is, (as expected) the Nash bargaining outcome is less favourable to the input supplier, and more favourable to the user, than is the tax rate t^*. In essence, the Nash solution removes the input supplier´s monopoly power when the tax rate is set.

It turns out that the Nash product is not everywhere concave, but it is concave around its maximum. This feature can be established by looking at the second derivative, but doing this is overly complex. Rather, we can look at the graph of the Nash product, and see its features there.

Using (12) together with (5), the Nash product is

$$\begin{aligned} N(t) &= \left(\frac{1}{4b}\right)\left(\frac{((1-t)-c)^2}{(1-t)}\right)\left(\frac{t((1-t)^2-c^2)}{4b(1-t)^2}\right) \\ &= \left(\frac{1}{4b}\right)^2\left(\frac{t((1-t)-c)^2((1-t)^2-c^2)}{(1-t)^3}\right) \end{aligned}$$

Since $(1-t)^2 - c^2 = ((1-t)-c)((1-t)+c)$, we can write the Nash product as

$$\begin{aligned} N(t) &= \left(\frac{1}{4b}\right)^2\left(\frac{t((1-t)-c)^2((1-t)-c)((1-t)+c)}{(1-t)^3}\right) \\ &= \left(\frac{1}{4b}\right)^2\left(\frac{((1-t)-c)}{1-t}\right)^3 t((1-t)+c) \qquad (14) \end{aligned}$$

Again, this is a rather complex looking equation, although it is certainly able to be analysed. The graph of the equation for $b = \frac{1}{4}$ and $c = \frac{1}{2}$ is displayed in Figure 4:

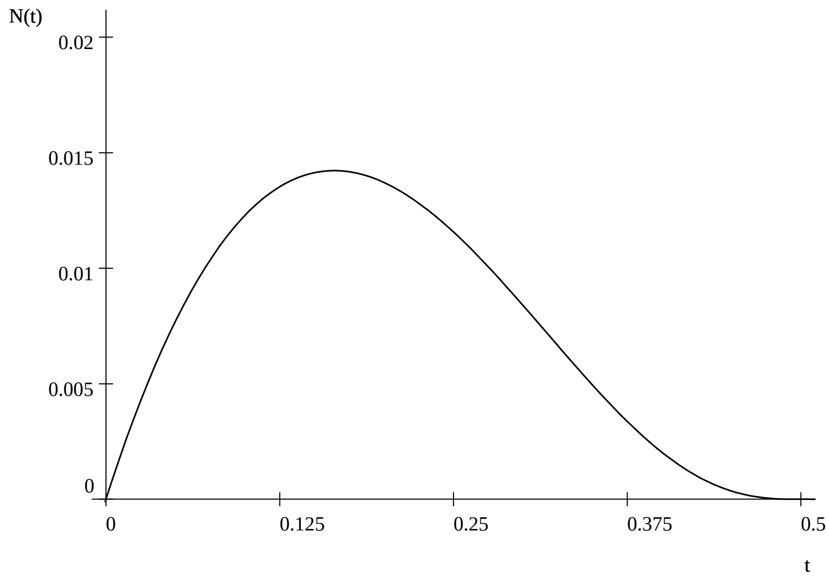

Figure 4: The graph of $N(t)$

Clearly, while not concave everywhere (specifically, it is not concave for values of t close to $\frac{1}{2}$), this graph has a local maximum at a strictly concave section of the graph. Indeed, $N(t)$ goes negative for values of t greater than $\frac{1}{2}$ (not shown in Figure 4) thus the maximum that is visable in the graph is the global optimum for the Nash product.

In order to locate the maximum of the Nash product, we need to find the t^N that solves $N'(t^N) = 0$. The first derivative of (14) can be expressed as[15]

$$N'(t) = \left(\frac{1}{4b}\right)^2 \left(\frac{1-t-c}{1-t}\right)^2 \left(\frac{1}{1-t}\right)^2 ((1-t-c)(1-2t+c)(1-t) - 3ct(1-t+c)) \tag{15}$$

As the first three terms are positive and finite, the first derivative is zero at

$$n(t,c) \equiv (1-t-c)(1-2t+c)(1-t) - 3ct(1-t+c) = 0$$

The graph of $n(t,c)$, for $c = \frac{1}{2}$ is drawn in Figure 5. The point at which it crosses the axis, i.e. the tax rate that maximises the Nash product, is $t^N = 0.16415$.

[15] The working is in section 7 of the appendix.

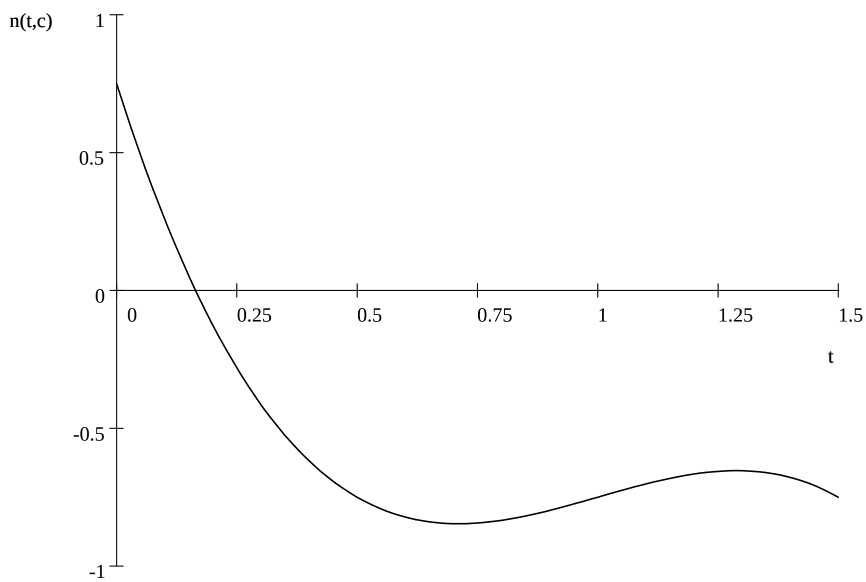

Figure 5: The shape of $n(t,c)$

Unfortunately, this time the algebraic solution to $N'(t^N) = 0$ proves to be extremely complex, and thus not particularly useful. Concretely, it turns out that the solution is given by

$$t^N(c) = \frac{1}{3}c + \sqrt[3]{k(c)} - \frac{\frac{2}{9}c^2 - \frac{2}{9}c - \frac{1}{36}}{\sqrt[3]{k(c)}} + \frac{5}{6}$$

where

$$k(c) \equiv \sqrt{\frac{1}{216}c^2 + \frac{13}{216}c^3 + \frac{19}{54}c^4 + \frac{1}{9}c^5 + \frac{1}{36}c^6} - \frac{5}{9}c^2 - \frac{7}{54}c^3 - \frac{1}{18}c - \frac{1}{216}$$

The tax rate $t^N(c)$ is graphed in Figure 6. It is strictly decreasing and convex.

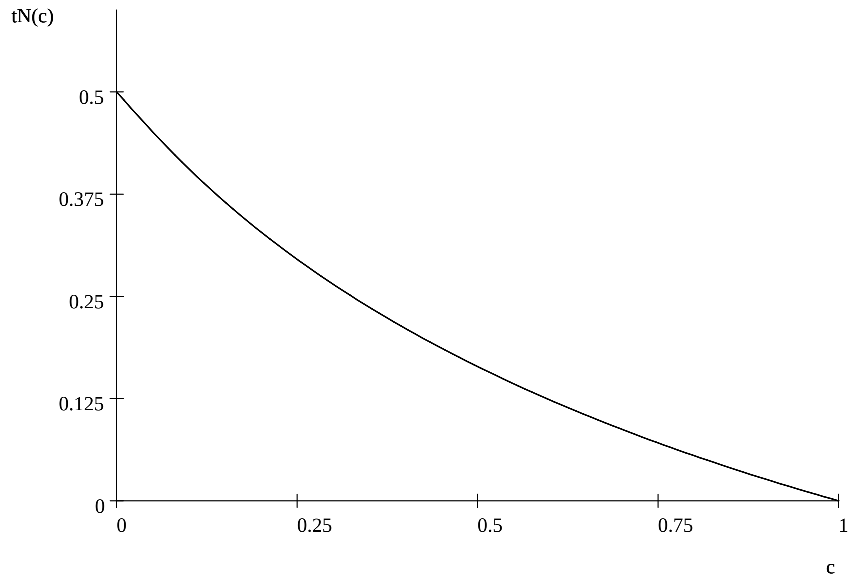

Figure 6: The Nash bargaining tax rate, t^N, as a function of c

The complexity of the algebraic form of $t^N(c)$ implies that it does not lend itself well to regulators. However, we can find a lower bound on t^N that does have a reasonably simple expression. Notice that $n(t,c)$ is convex to the left of the point t^N. Given that, the curve $n(t,c)$ lies above the tangent to this curve at $t=0$. This implies that the value of t^N must always be greater than the point at which the tangent line to $n(t,c)$ at $t=0$ cuts the horizontal axis. Lets call that point t^{Ne}, since it is a value that estimates t^N.

The derivative of $n(t,c)$ is

$$\frac{\partial n(t,c)}{\partial t} = (1-t)(6t-4) + c(4t-2) - 2c^2$$

At $t=0$, this is

$$\left.\frac{\partial n(t,c)}{\partial t}\right|_{t=0} = -4 - 2c - 2c^2$$

Also, note that $n(0,c) = (1-c)(1+c) = 1-c^2$. So, the point at which the tangent line cuts the horizontal axis is given by

$$\begin{aligned} t^{Ne} &= \frac{1-c^2}{4+2c+2c^2} \\ &= \left(\frac{1}{2}\right)\left(\frac{1-c^2}{2+c+c^2}\right) \end{aligned} \tag{16}$$

For example, with $c = \frac{1}{2}$, for which $t^N = 0.16415$, we get $t^{Ne} = 0.13636$. In Figure 7, the straight line shows the tangent to $n(t, c)$ at $t = 0$, and the point at which that line cuts the axis is t^{Ne}, which clearly lies to the left of t^N.

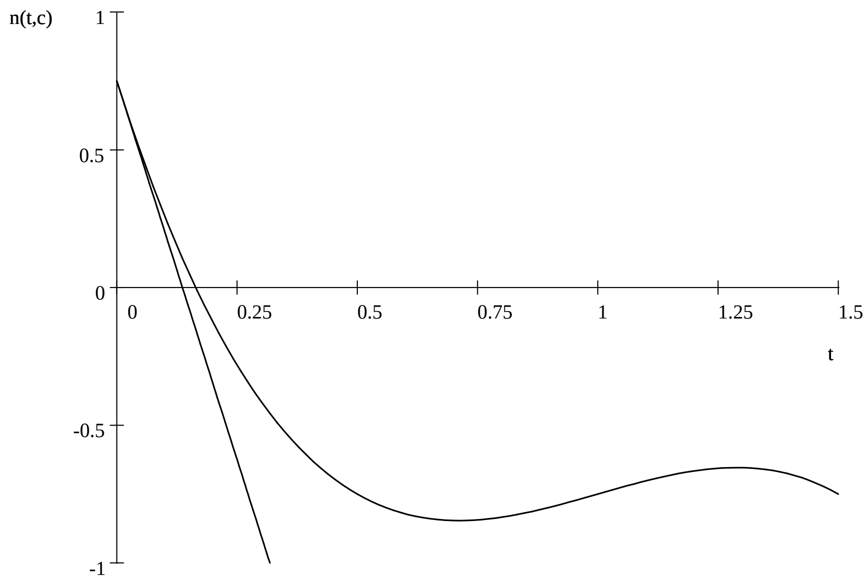

Figure 7: The relationship between t^N and t^{Ne}

The graph of the lower bound on the Nash tax rate, as a function of c, is given in Figure 8 (the lower curve), along with the Nash rate (the higher curve). The graph of $t^{Ne}(c)$ is a strictly decreasing and concave function.[16]

[16] The shape of this function suggests an even easier, although slightly worse approximation. Since $t^{Ne}(c)$ is concave, it lies everywhere above the straight line through its endpoints. This straight line is $t = \frac{1}{4}(1 - c)$. Thus, $\frac{1}{4}(1 - c) \leq t^{Ne}(c)$ for all c. Since $t^{Ne}(c)$ is not very concave, the approximation should be reasonably good, and clearly the equation $\frac{1}{4}(1-c)$ is extremely simple. For the case of $c = \frac{1}{2}$, for which $t^N = 0.16415$, and $t^{Ne} = 0.13636$, we get $\frac{1}{4}(1 - c) = 0.125$.

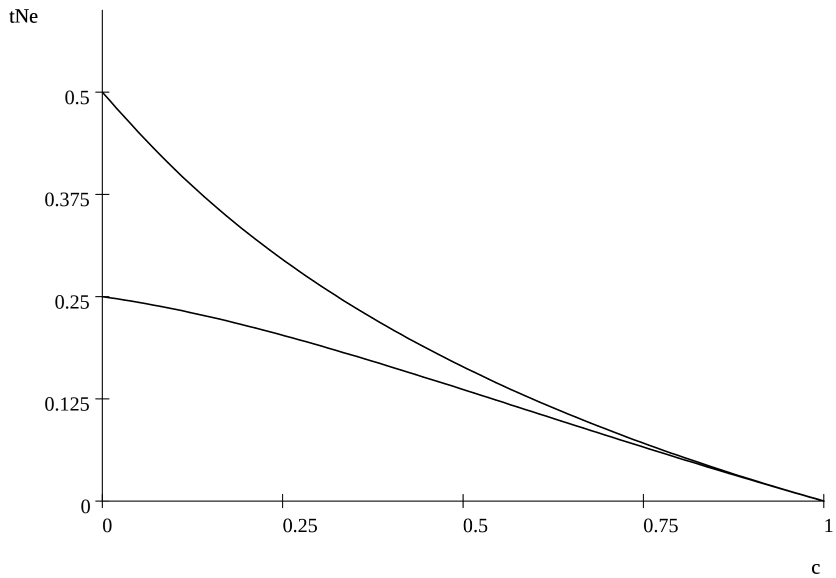

Figure 8: The shape of $t^{Ne}(c)$ in comparison with $t^N(c)$

From Figure 8, we can see that $t^{Ne}(c)$ better approximates $t^N(c)$ the greater is c. It is also evident that, given the shape of $t^N(c)$, other approximations are surely feasible, so $t^{Ne}(c)$ should not be thought of as the only approximating lower bound function. For example, it happens that the slope[17] of the function $t^N(c)$ is -1 at the point $c = 0$, and the slope is $-\frac{1}{4}$ at the point $c = 1$. Thus, the function $t^N(c)$ lies everywhere above the two straight lines

$$\begin{aligned} t &= \frac{1}{2} - c \\ t &= \frac{1}{4}(1 - c) \end{aligned}$$

These two lines, along with $t^N(c)$ are shown in Figure 9.

[17] Of course, evaluating the slope of $t^N(c)$ is very complex. I used Matlab to evaluate the derivative, and to calculate its value at the endpoints.

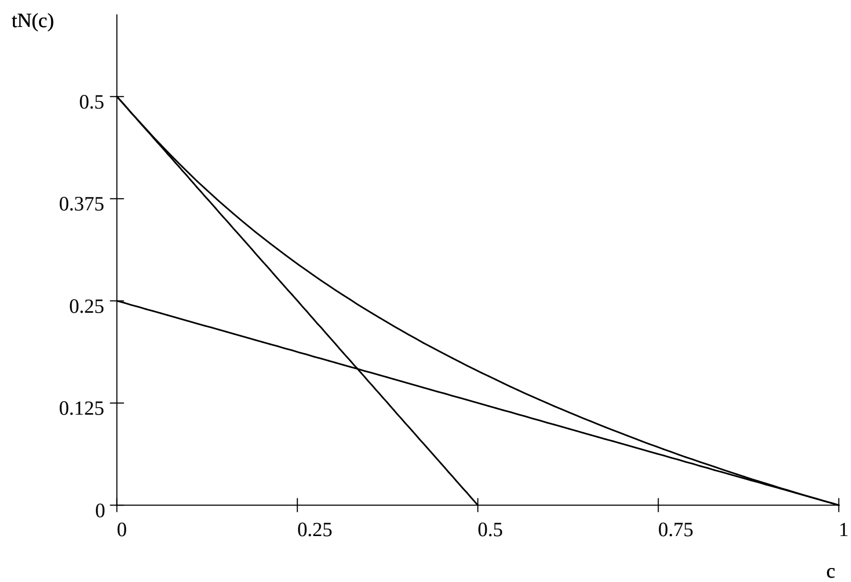

Figure 9: The Nash bargaining tax rate, t^N, with two of its supporting tangents

The two supporting tangent lines intersect at $c = \frac{1}{3}$, at which point $t = \frac{1}{6}$. Given this, consider the upper envelope of the two straight lines, that is, the piecewise function

$$t^E(c) = \begin{cases} \frac{1}{2} - c & \text{for } 0 \le c \le \frac{1}{3} \\ \frac{1}{4}(1-c) & \text{for } \frac{1}{3} \le c \le 1 \end{cases}$$

Clearly, $t^E(c)$ also provides a lower bound on the Nash bargaining tax rate. This rate is a better approximation to the Nash rate for low values of c, but a slightly worse approximation for high values of c. Of course, we could also look for the upper envelope of the two functions $t = \frac{1}{2} - c$ and $t^{Ne}(c)$, which would give us a better approximation than just $t^{Ne}(c)$ for low values of c and the same approximation for high values of c. Any number of other options can be devised as well. However, since it seems a little arbitrary to choose one option over any other, in what follows I shall simply stick with $t^{Ne}(c)$ as the lower bound approximation to the Nash solution.

We can consider the way in which the total surplus is shared when the lower bound to the Nash solution is used. To do that, we only need to substitute (16) into (11). This gives us the equation[18]

$$\frac{\pi(t^{Ne})}{R(t^{Ne})} = \left(\frac{3+4c+5c^2}{1-c^2}\right)\left(\frac{3+3c^2-2c^3}{3+8c+7c^2+2c^3}\right)$$

[18] See section 8 of the appendix for the working.

Again, a rather formidable looking expression, but one that can be easily graphed. The graph of the surplus sharing regime under the lower bound tax rate is given in Figure 10. The curve is everywhere convex, valued at 3 when $c = 0$, goes infinite as $c \to \infty$, and has a minimum at c approximately equal to 0.2055. At the minimum of the curve, the value of $\frac{\pi}{R}$ is clearly greater than 2.5 (actually, its minimum value is about 2.6414), and so for this tax rate, the user always gets the lion's share of the total surplus (the user's profit is at least two and a half times greater than the input supplier's income for all possible values of c).

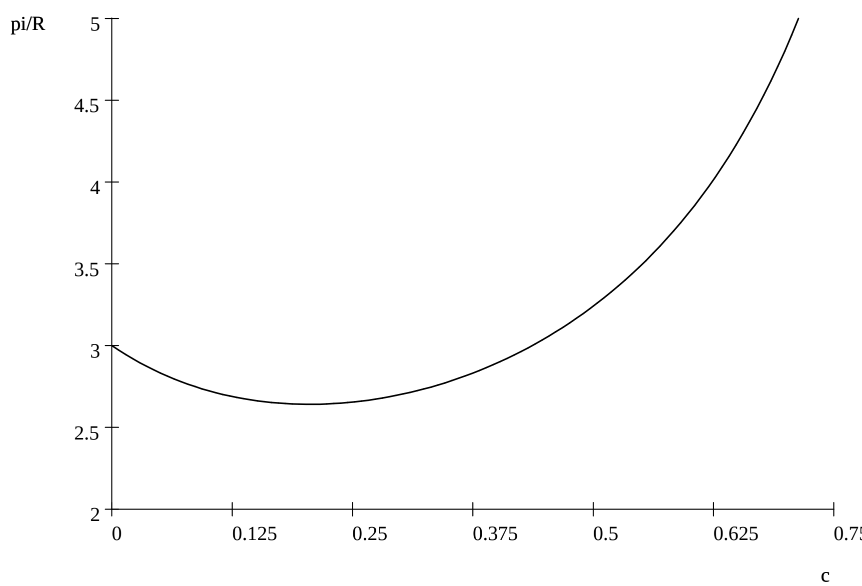

Figure 10: The shape of $\frac{\pi(t^{Ne})}{R(t^{Ne})}$ for small values of c

6 Fixed costs

One concern with the modelling above is the assumption that the user produces with no fixed costs. In reality, this assumption does not alter the way in which either the input supplier's optimal revenue tax, or the Nash bargaining revenue tax are calculated. Of course, it also does not affect the calculation of the user's optimal output, unless it serves to set optimal output to 0. The optimal revenue tax is simply found where the input supplier's income is maximised, which is independent of any fixed costs in the user's production process. In the Nash model, in reality the Nash product is

$$N(t) = \left(R(t) - \overline{R}\right)\left(\pi(t) - \overline{\pi}\right)$$

where $\overline{R}$ and $\overline{\pi}$ are, respectively, the earnings of the input supplier and the user when no deal is struck. When fixed costs are 0, since no deal means no surplus to share, we get $\overline{R} = \overline{\pi} = 0$. When there are fixed costs of F, we still have $\overline{R} = 0$, but now we have $\pi_F(t) = \pi(t) - F$, and $\overline{\pi} = -F$. Therefore, we would have

$$\begin{aligned} N(t) &= \left(R(t) - \overline{R}\right)\left(\pi_F(t) - \overline{\pi}\right) \\ &= R(t)(\pi(t) - F + F) \\ &= R(t)\pi(t) \end{aligned}$$

The *only* effect of including fixed costs for the user is that the upper limit revenue tax rate that can be charged is altered, and may end up being a binding constraint upon the problem.

Specifically, since we know that $\left[(1-\bar{t})(1-bx^*(\bar{t})) - c\right] x^*(\bar{t})$ is decreasing in the tax rate, when the user faces fixed costs of F, the tax rate must be set such that

$$\pi_F(t) = \left[(1-t)(1-bx^*(t)) - c\right] x^*(t) - F \geq 0$$

Using (12), the requirement is that

$$\left(\frac{1}{4b}\right)\left(\frac{(1-t-c)^2}{(1-t)}\right) \geq F$$

that is

$$(1-t-c)^2 - (1-t)4bF \geq 0$$

Consider the function $h(t) \equiv (1-t-c)^2 - (1-t)4bF$. Its slopes are

$$\begin{aligned} h'(t) &= -2(1-t-c) + 4bF \\ h''(t) &= 2 > 0 \\ \frac{\partial h}{\partial F} &= -(1-t)4b < 0 \end{aligned}$$

Since its second derivative is positive, $h(t)$ is a convex function. Since the derivative in F is negative, an increase in F shifts the function downwards.

Let's start with the case of $F = 0$. In that case, $h(t)$ reaches its minimum exactly at the horizontal axis. The graph in question is shown in Figure 11, in which $c = \frac{1}{2}$. The higher graph is that corresponding to $F = 0$. At that point, we have $t = 1 - c$. We are only concerned with the decreasing part of the function, since we know that we must always restrict $t \leq 1 - c$ in order that output be non-negative. In the case of $F = 0$, the entire negatively sloped part of the function is valued positive, and so (as we already saw above), in this case the only restriction on t is that it cannot go above $1 - c$.

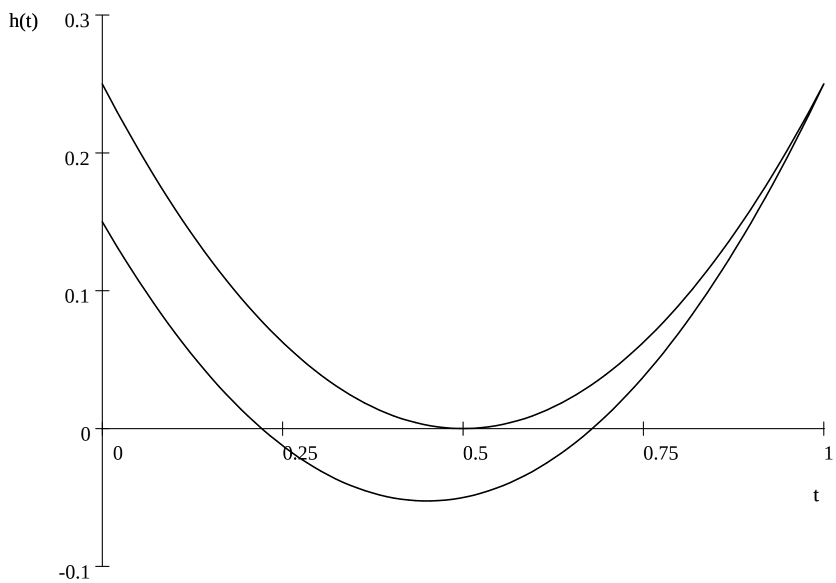

Figure 11: The shape of $h(t)$ for $F = 0$ (higher graph) and for $F > 0$ (lower graph).

Now, as F goes positive, the function moves downwards. This will generate two roots of the function, one above $1 - c$ (where the function will have positive slope) and one below $1 - c$ (where the function will have negative slope). Such a case is drawn as the lower graph in Figure 11, where it is assumed that $c = \frac{1}{2}$, $bF = \frac{1}{40}$.

Since the function $h(t)$ is negative between the lower root and the value $1 - c$, we know that we can only consider as valid tax rates those that are no greater than the lower root. Using the quadratic formula, it can be shown that the lower root is at

$$t_0(c, F) = (1 - c - 2bF) - 2\sqrt{bF}\sqrt{bF + c} \tag{17}$$

Now, note that if the revenue tax were to be set at 0, the user receives the input free of charge. It is not conceivable that this would result in a negative overall profit. If it did, then really there is no reasonable business proposal in the first place. Thus, at $t = 0$, it is entirely reasonable that the user earns a strictly positive profit. In terms of our graphs, this converts to the condition that $h(0) > 0$, that is

$$1 - 4bF - 2c - c^2 > 0$$

or

$$\left(\frac{1}{4b}\right)\left(1 - 2c - c^2\right) > F$$

Note that this condition implies that $t_0(c, F) > 0$, i.e. there is always something that can be shared.

The graph of $t_0(c, F)$, for $bF = \frac{1}{40}$ is displayed in Figure 12. It is a strictly decreasing and (slightly) convex function.

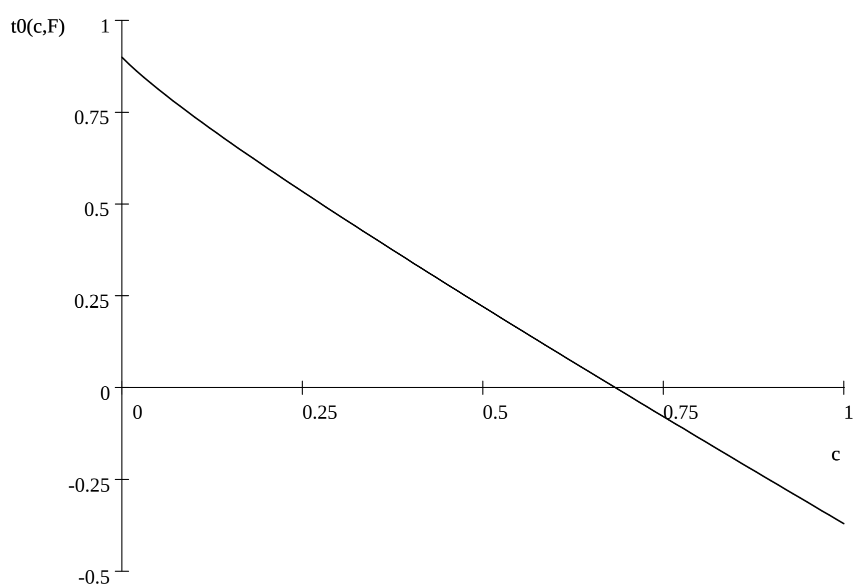

Figure 12: The limit tax rate as a function of c, with positive fixed costs

Notice that, for $bF = \frac{1}{40}$, the limit tax rate is only positive for low enough values of c. In fact, with this example, the limit tax rate goes to 0 at $c = 0.68377$. All this is really saying is that when the marginal cost is high enough and a fixed cost is present, the user's business becomes unprofitable, since even if the input were supplied free of charge, the user still makes a non-positive profit.

As F goes up, the graph of the limit tax rate moves downwards. This clearly implies that, if it is intended that the lower bound on the Nash tax rate should be used, there is an upper bound on the level of fixed costs that can be present. Specifically, recall that the graph of $t^{Ne}(c)$ is strictly negatively sloped, and $t^{Ne}(0) = \frac{1}{4}$. It also happens that $t^{Ne}(c)$ is a flatter function than is $t_0(c, F)$.[19] Thus, if this lower bound tax rate is ever to be feasible, we require that the limit tax rate at $c = 0$ be no smaller than $\frac{1}{4}$, otherwise the limit tax rate will be smaller than the lower bound rate for all c. The restriction then is that $t_0(0, F) = (1 - 2bF) - 2\sqrt{bF}\sqrt{bF} = 1 - 4bF \geq \frac{1}{4}$. With a very mininal amount of effort, this restriction can be expressed as

$$bF \leq \frac{3}{16} \tag{18}$$

Of course, even if restriction (18) is satisfied, there is no *guarantee* that the lower bound on the Nash solution tax rate is feasible. It depends upon the value of the marginal cost c. The restriction (18) is thus a necessary, but not sufficient, condition for the lower bound on the Nash tax rate to be feasible.

Finally, in Figure 13 we graph the limit tax rate (in red) together with the optimal tax rate (in black) and the lower bound on the Nash solution tax rate (in purple). While not really visible in the graph, for the parameters chosen, it turns out that the optimal rate is actually lower than the limit rate, but only for a very small range of value of c. Concretely, between about $c = 0.1162$ and $c = 0.2$, the optimal rate dips below the limit rate (although only very very slightly). Thus (at least for the parameter values used here), the fixed cost makes the optimal tax rate impossible for all but a very small range of values of c. Since the limit tax function shifts downwards with an increase in F, we can easily see that there will be values of F (not significantly greater than the $\frac{1}{10}$ that is assumed in the graph) for which there would be no values of c for which the optimal

[19] The slope of $t^{Ne}(c)$ can be calculated as $-\left(\frac{1}{c+c^2+2}\right)\left(c + \frac{(2c+1)(1-c^2)}{c+c^2+2}\right)$. We also know that $t^{Ne}(c)$ is a decreasing concave function, thus it takes its smallest slope (i.e. its steepest part) at $c = 1$, where its slope is $-\frac{1}{4}$. Thus, the slope of $t^{Ne}(c)$ is everywhere greater than $-\frac{1}{4}$. On the other hand, the slope of $t_0(c, F)$ in c turns out to be $-1 - \sqrt{\frac{bF}{bF+c}}$ which is clearly less than -1 for all c. Thus, for all c the function $t_0(c, F)$ takes a smaller negative slope (i.e. is steeper) than the function $t^{Ne}(c)$.

tax rate is feasible. However, the lower bound on the Nash tax rate is lower than the limit tax for all but quite large levels of c. Thus the lower bound on the Nash tax rate is feasible for a large set of values of marginal cost. Concretely, with the parameters used in this example, the lower bound on the Nash tax rate is below the limit tax rate for all values of c less than 0.59116.

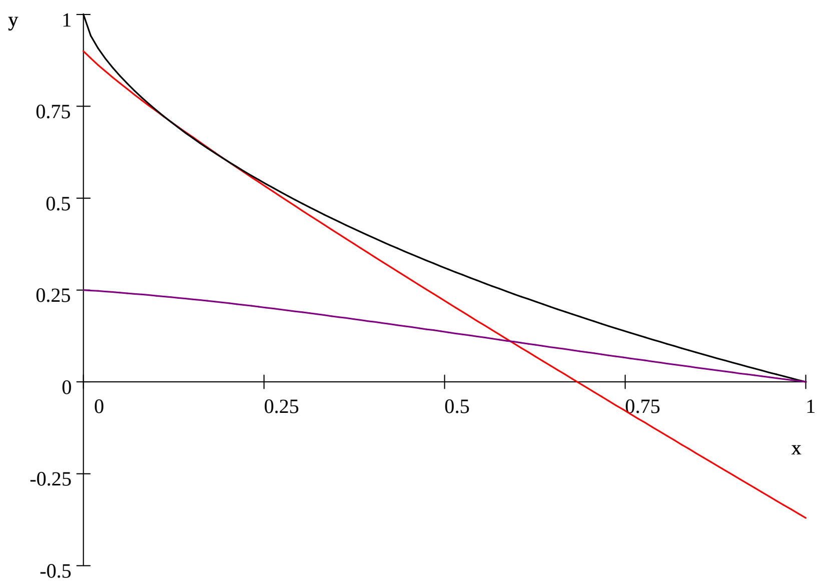

Figure 13: Comparison between the limit tax rate, the optimal tax rate, and the lower bound Nash tax rate

7 Regulatory policy recommendations

While simplified with respect to the real world, the above model allows us to grasp some initial insights into essential input contracts that are stipulated to be a revenue sharing arrangement between the user and the input supplier. The model looks only at optimal sharing arrangements from the perspective of the input supplier,[20] although we have also established an interesting lower bound on the tax rate that would emerge from an evenly structured bargaining game. The model only requires knowledge of a few concrete variables in order for it to provide guidance as to the optimal tax rate, and of course as to the lower bound on the bargaining tax rate. Specifically,

[20] Of course, an optimal arrangement from the perspective of the user is to simply set the tax rate to 0.

we need to know the value of c, the marginal cost as a fraction of the vertical intercept of the demand function.[21] If there is a fixed cost present in the user's production process, then F would also have to be known, since the fixed cost alters the set of feasible tax rates that can be used, as would the slope parameter of the demand curve.

The most interesting rate from the perspective of regulatory policy is the lower bound on the Nash bargaining rate. The Nash bargaining model posits an unrestricted bargaining process, with equal bargaining powers. Thus the rate that emerges from the Nash bargaining game is fair and equitable to all parties. When compared to the Nash solution, the lower bound that we have established for the Nash rate is more favourable to the user than to the input supplier. It also happens that the lower bound on the Nash rate is likely to be feasible when there is a fixed cost present in the problem, at least for a reasonably large set of parameters. Thus, the lower bound on the tax rate may go some distance in solving the problem that is often faced by regulators when they attempt to set a fair revenue tariff.

Given that there are likely to be some fixed costs in the user's production function, my suggestion for a regulated revenue tax is

$$t_r = \min\left\{\left(\frac{1}{2}\right)\left(\frac{1-c^2}{2+c+c^2}\right), (1-c-2bF) - 2\sqrt{bF}\sqrt{bF+c}\right\} \tag{19}$$

We can use this to get an idea about how well reglators in the real world have managed to set a fair rate. In order to do this, I shall posit a simple example, based on the broadcast music radio business. In the music radio business, where a regulated revenue tax is habitually used, this tax rate is normally held under 5%, although there are cases in which a rate as high as 10% is levied on some radio stations.[22]

Consider the traditional radio broadcasting business. The radio station requires music in order to function, and the music is supplied as an entire repertory by the copyright holders acting together as a copyright collective. The good produced, x, is audience which is then sold to advertisers. Since advertisers will have decreasing marginal willingness to pay for an aditional

[21] If, in a real-world setting, the demand function and/or the cost function were estimated to be non-linear, it would be necessary to simply extract the linear approximation to each curve in order to apply the model.

[22] This is the case, for example, in Ireland and India, for radio stations that are particularly high users of music as an input.

unit of audience, it is certainly reasonable that the demand curve for advertising faced by a radio station is decreasing in the audience size, just as has been assumed in this paper. Thus, it is reasonable to assume that there is indeed some price at which demand would go to 0 (at least, in a linear approximation to the demand curve).

Consider the cost situation of a typical radio station. The radio business is largely run according to a cost model that is independent of the audience reached.[23] The main costs are to obtain frequency, purchasing and maintaining broadcasting equipment, office space, wages of disc-jockies, program directors and executives, etc.[24] In fact the only cost that would be associated with altering the audience would seem to be those costs incurred in marketing.

Take a radio station operating in a medium sized city. Assume that the demand curve for advertising is given by[25]

$$p = 1 - \frac{x}{8,000,000}$$

Say that outside of any payment for the music input (i.e. for now we set $t = 0$), the station has \$1.5 million in total costs. Let's say frequency costs, equipment, office space, wages etc. (i.e. costs that are independent of audience) all add up to \$1.3 million, leaving \$200,000 per year for marketing (the only variable cost). We know that the station will set output (i.e. audience)[26] according to (3), which with $t = 0$ is

$$\frac{1-c}{2b} = \frac{8,000,000(1-c)}{2} = 4,000,000(1-c)$$

The average variable cost (per unit audience) is thus equal to

$$\frac{200,000}{4,000,000(1-c)} = 0.05(1-c)$$

[23] There is a difference here with internet broadcast radio. When more audience is to be reached, greater bandwidth is required, and so there is a strong element of variable cost in the internet radio business.

[24] Of course the radio must also pay collecting organisations for the right to broadcast music. However, since this is exactly the fee that we are interested in calcluating here, we shall deal with it separately.

[25] Here, x is the number of listeners of the station, averaged over all broadcast minutes.

[26] Of course, in the real world of music radio, the station may not be able to easily pick and choose its audience level. The example is only meant to be illustrative. If the audience achieved is not the optimal level, then the station will simply earn less revenue and profit than would be feasible in the optimum. This will typically mean that the restriction on the maximum revenue sharing tax is reached earlier. The example can be easily modified to fit the case of sub-optimal audience levels.

If the cost function is indeed linear, then average variable cost is equal to marginal cost, so

$$c = 0.05(1 - c)$$

from which $c = 0.0476$.

Now, we also know that the profit in general is given by

$$\pi = \left(\frac{1}{4b}\right)\left(\frac{(1-t-c)^2}{(1-t)}\right) - F$$

Substituting in the numbers that we have ($t = 0$, $b = \frac{1}{8{,}000{,}000}$, $c = 0.0476$, and $F = 1,300,000$), we get

$$\begin{aligned}
\pi &= \frac{8,000,000(1-0.0476)^2}{4} - 1,300,000 \\
&= 2,000,000(1-0.0476)^2 - 1,300,000 \\
&= 514,131.52
\end{aligned}$$

The lower bound on the Nash tax rate is

$$\left(\frac{1}{2}\right)\left(\frac{1-c^2}{2+c+c^2}\right) = \left(\frac{1}{2}\right)\left(\frac{1-0.0476^2}{2+0.0476+0.0476^2}\right) = 0.24337$$

And, since $bF = \frac{1{,}300{,}000}{8.000.000} = 0.1625$ the limit tax rate is

$$\begin{aligned}
(1-c-2bF) - 2\sqrt{bF}\sqrt{bF+c} &= (1-0.0476-0.325) - 2\sqrt{0.1625}\sqrt{0.1625+0.0476} \\
&= 0.25785
\end{aligned}$$

Since the limit rate is greater than the Nash rate, the Nash rate is a feasible tax (i.e. it leaves the user with strictly positive profit). Thus, with this example, the regulator should set the revenue tax rate at about 24.3%. At that tax rate, the user still earns strictly positive profit, and the rate is, by definition, fair to both parties.[27] For comparison, in this example the value of $\frac{1}{4}(1-c)$, which as we saw earlier is a value that approximates (but is less than) the lower bound on the Nash tax rate when $c > \frac{1}{3}$, is 0.2381.

[27] Since this is only a ficticious example, we should not read too much into the large diference between the revenue tax rate that the model gives (24.3%) and the real-world rates of closer to 5%. Until real-world data can be brought to the model to calibrate it, we cannot know how the 5% standard compares to a fair negotiated rate.

At the tax rate of 0.243, using (12) and (5) we can calculate that the total profit of the user is $\$29,586.16$, and the copyright collective earns income of $\$484,078,42$. It is, perhaps, instructive to note that, when the revenue tax is installed, the total surplus is $29,586.16 + 484,078,42 = 513,664.58$, which is smaller than the total surplus when the tax is set at 0 (which is $514,131.52$). Thus, it would be better for the copyright collective to set a tax rate of 0, and to charge a fixed fee of $514,131.52 - 29,586.16 = 484,545.36$. If this could be done, the user would be indifferent to the fixed fee situation and the regulated revenue tax, but the copyright collective would be better off. This example serves to show that although in many situations the regulator makes a revenue tax the compulsory compensation mechanism, this may be inefficient.

8 Conclusions

In this paper I have analysed a model in which an input supplier supplies an essential input to a user, and in exchange for the input the input supplier is remunerated under a revenue sharing rule, that is, by a tax on the user's revenue. The analysis shows that, if there are no fixed costs in the user's production process, the optimal tax rate (from the point of view of the input supplier) will always leave the user with positive earnings, but in most situations (those with a strictly positive marginal cost) will imply that the input supplier earns more of the total surplus that is generated than does the user.

If the monopoly power that the input supplier has when the remuneration system is negotiated is considered to be excessive, then one could look to the Nash bargaining model for guidance as to a fair revenue tax. The Nash bargaining model removes any monopoly power that the input supplier has over the user when the revenue tax is set, and thus it constitutes a fair tariff. It turns out that while the Nash tax rate can always be calculated (once the parameters of the model are known), it is excessively complex. However, an interesting lower bound on the Nash tax rate can be found, and it has a relatively simple expression. Using the lower bound on the Nash tax rate, if the user's production process has no fixed costs, then the user will always get a significantly greater share of the total surplus than would the input supplier.

In the paper we have looked at both the simple case in which the user operates with no fixed costs, and the more complex case in which fixed costs are present. The presence of fixed costs does not alter the calculation of either the optimal revenue tax rate or the Nash tax rate (or the lower bound on the Nash tax rate), but it does imply that there is an upper bound on the tax rate that can be used. However, we have shown (by example) that while the upper bound might be a serious impediment to the use of an optimal tax rate, it generally appears to leave plenty of scope for the use of the lower bound on the Nash tax rate.

It is worthwhile to point out that, if a regulator is interested in devising a revenue tax as remuneration for an essential input, it is of great aid that the formula for the tax be as simple and user-friendly as possible. The tax suggested in this paper seems to satisfy these requirements. In order to calculate the lower bound on the Nash tax rate, the regulator only requires to know the levels of marginal and fixed costs, and the vertical intercept of the demand curve for the producer's output. This vertical intercept could, of course, more realistically be found in a linear approximation to the demand curve.

The model in the present paper is only put forward as a first step, and it can be improved upon and more generally modified in several interesting ways.

The most obvious extention to the present model would be to calibrate it against some real-world data. While I do not have any hard data on hand concerning the operation of music radio generally, I conjecture that the lower bound on the Nash tax rate suggested in this paper is significantly greater than the tariffs that are levied in the real world as a result of regulatory practices in music radio. Since the lower bound is based upon a fair and unconstrained bargaining game, it removes any monopoly power held by the input supplier, I conjecture that the regulated tariffs are generally too low. However, I stress that this is only conjecture, and the true relationship between the real-world regulated rates and the fair rate suggested here can only be known once some real-world data is inserted into the present model.

At a theoretical level, it is interesting to wonder what would happen if instead of supplying the input to a single user, the input supplier were to deal with many users? That is, what if the user actually operated in an oligopolistic market, rather than being a monopolist. So long as the input supplier were obliged not to refuse any user (compulsory licensing), and were obliged not to discriminate amoung users in terms of price, the model might not change much from what is present in the current paper. Still, it would be interesting to find out.

Appendix

In several places in the following proofs, we need to use the value of the market price of the user´s good, $1 - bx^*(t)$. Since we have

$$x^*(t) = \frac{(1-t)-c}{2b(1-t)}$$

it turns out that

$$\begin{aligned} 1 - bx^*(t) &= 1 - b\left(\frac{(1-t)-c}{2b(1-t)}\right) \\ &= 1 - \left(\frac{(1-t)-c}{2(1-t)}\right) \end{aligned}$$

Giving this a common denominator, we get

$$\begin{aligned} 1 - bx^*(t) &= \frac{2(1-t)-(1-t)+c}{2(1-t)} \\ &= \frac{(1-t)+c}{2(1-t)} \end{aligned} \tag{20}$$

1) In Theorem 1, we are interested in finding the tax rate that sets $(1-\bar{t}_2)(1-bx^*(\bar{t}_2))-c=0$. Substituting for the price from (20), we get

$$(1-\bar{t}_2)\left(\frac{(1-\bar{t}_2)+c}{2(1-\bar{t}_2)}\right) = c$$

Cancelling the $(1-\bar{t})$ that appears now in both the numerator and denominator of the left-hand-side gives

$$\frac{(1-\bar{t}_2)+c}{2} = c$$

which simplifies directly to the equation given in the text.

2) To find the input supplier's income function, firstly substitute (3) and (20) into (2);

$$\begin{aligned} R(t) &= t(1-bx^*(t))x^*(t) \\ &= t\left(\frac{(1-t)+c}{2(1-t)}\right)\left(\frac{(1-t)-c}{2b(1-t)}\right) \end{aligned}$$

Combining the terms in both the numerator and denominator we get

$$\begin{aligned} R(t) &= \frac{t((1-t)+c)((1-t)-c)}{4b(1-t)^2} \\ &= \frac{t((1-t)^2-c^2)}{4b(1-t)^2} \end{aligned}$$

which is the equation given in the text.

3) The working necesary to find the derivatives of $R(t)$ is the following. Firstly, write $R(t)$ as

$$R(t) = \left(\frac{1}{4b}\right)\left(\frac{t}{1-t}\right)\left(\frac{(1-t)^2-c^2}{1-t}\right)$$

Then, using the quotient rule, we get

$$\begin{aligned} R'(t) &= \left(\frac{1}{4b}\right)\left[\left(\frac{1}{(1-t)^2}\right)\left(\frac{(1-t)^2-c^2}{1-t}\right)+\left(\frac{t}{1-t}\right)\left(\frac{-2(1-t)(1-t)+((1-t)^2-c^2)}{(1-t)^2}\right)\right] \\ &= \left(\frac{1}{4b}\right)\left(\frac{1}{(1-t)^3}\right)\left((1-t)^2-c^2+t\left(-2(1-t)^2+(1-t)^2-c^2\right)\right) \\ &= \left(\frac{1}{4b}\right)\left(\frac{1}{(1-t)^3}\right)\left((1-t)^2-c^2+t\left(-(1-t)^2-c^2\right)\right)\Big) \\ &= \left(\frac{1}{4b}\right)\left(\frac{1}{(1-t)^3}\right)\left((1-t)^2(1-t)-(1+t)c^2\right) \\ &= \left(\frac{1}{4b}\right)\left(\frac{1}{(1-t)^3}\right)\left((1-t)^3-(1+t)c^2\right) \\ &= \left(\frac{1}{4b}\right)\left(1-\frac{(1+t)c^2}{(1-t)^3}\right) \end{aligned}$$

This is the equation given in the text.

To find the second derivative, again we use the quotient rule;

$$R''(t) = -\left(\frac{1}{4b}\right)\left(\frac{c^2(1-t)^3+(1+t)c^2 3(1-t)^2}{(1-t)^6}\right)$$

Cancelling the $(1-t)^2$, we get

$$R''(t) = -\left(\frac{1}{4b}\right)\left(\frac{c^2(1-t)+(1+t)c^2 3}{(1-t)^4}\right)$$

Again, we only need to simplify the numerator. Taking out the common factor, c^2, and expanding the numerator gives

$$R''(t) = -\left(\frac{c^2}{4b}\right)\left(\frac{4+2t)}{(1-t)^4}\right)$$

Finally, dividing top and bottom by 2 gives the equation in the text

$$R''(t) = -\left(\frac{c^2}{2b}\right)\left(\frac{2+t)}{(1-t)^4}\right)$$

4) To express the user's profit as a function of the input supplier's income, we firstly split the profit equation into three separate terms;

$$\begin{aligned}\pi(t) &= \left[(1-t)(1-bx^*(t)) - c\right]x^*(t) \\ &= (1-bx^*(t))x^*(t) - t(1-bx^*(t))x^*(t) - x^*(t)c\end{aligned}$$

But, the second of those terms is equal to the input supplier's income, and the first term is input supplier income divided by t;

$$\pi(t) = \frac{R(t)}{t} - R(t) - x^*(t)c$$

Joining the first two terms gives the equation in the text;

$$\pi(t) = R(t)\left(\frac{1-t}{t}\right) - x^*(t)c$$

5) In order to get the reduced form for the ratio of profit to input supplier income, substituting the optimal output level into (10) to get

$$\frac{\pi(t)}{R(t)} = \left(\frac{1-t}{t}\right) - \frac{c}{t\left[1 - b\left(\frac{(1-t)-c}{2b(1-t)}\right)\right]}$$

The square bracket term is just the price, so substitute in from (20) to get

$$\begin{aligned}\frac{\pi(t)}{R(t)} &= \left(\frac{1-t}{t}\right) - \frac{c}{t\left(\frac{(1-t)+c}{2(1-t)}\right)} \\ &= \left(\frac{1-t}{t}\right) - \left(\frac{c}{(1-t)+c}\right)\left(\frac{2(1-t)}{t}\right)\end{aligned}$$

Now we see that there is a common factor of $\frac{1-t}{t}$, which we extract to get

$$\frac{\pi(t)}{R(t)} = \left(\frac{1-t}{t}\right)\left[1 - \frac{2c}{(1-t)+c}\right]$$

Finally, giving the square bracket term a common denominator and simplifying gives the equation in the text;

$$\begin{aligned}\frac{\pi(t)}{R(t)} &= \left(\frac{1-t}{t}\right)\left[\frac{(1-t)+c-2c}{(1-t)+c}\right] \\ &= \left(\frac{1-t}{t}\right)\left[\frac{(1-t)-c}{(1-t)+c}\right]\end{aligned}$$

6) The user's profit is

$$\pi(t) = \left(\frac{1}{4b}\right)\left(\frac{(1-t-c)^2}{1-t}\right)$$

Using the quotient rule, the derivative of this is

$$\pi'(t) = \left(\frac{1}{4b}\right)\left(\frac{-2(1-t-c)(1-t)+(1-t-c)^2}{(1-t)^2}\right)$$

Taking out the common factor in the second bracketed term, this is

$$\begin{aligned}
\pi'(t) &= \left(\frac{1}{4b}\right)\left(\frac{(1-t-c)(-2(1-t)+(1-t-c))}{(1-t)^2}\right)\\
&= \left(\frac{1}{4b}\right)\left(\frac{(1-t-c)(-2+2t+1-t-c)}{(1-t)^2}\right)\\
&= \left(\frac{1}{4b}\right)\left(\frac{(1-t-c)(t-1-c)}{(1-t)^2}\right)\\
&= -\left(\frac{1}{4b}\right)\left(\frac{(1-t-c)(1-t+c)}{(1-t)^2}\right)
\end{aligned}$$

But since $(1-t-c)(1-t+c) = (1-t)^2 - c^2$, we end up with

$$\begin{aligned}
\pi'(t) &= -\left(\frac{1}{4b}\right)\left(\frac{(1-t)^2-c^2}{(1-t)^2}\right)\\
&= -\frac{R(t)}{t}
\end{aligned}$$

where we have used (5).

7) To find the derivative of the Nash product, we derive (14) using the quotient rule;

$$N'(t) = \left(\frac{1}{4b}\right)^2\left[3\left(\frac{1-t-c}{1-t}\right)^2\left(\frac{-c}{(1-t)^2}\right)t(1-t+c)+\left(\frac{1-t-c}{1-t}\right)^3(1-2t+c)\right]$$

Multiply the second term in square brackets by $\frac{1-t}{1-t}$, and then take out the common factors;

$$N'(t) = \left(\frac{1}{4b}\right)^2\left(\frac{1-t-c}{1-t}\right)^2\left(\frac{1}{(1-t)^2}\right)\left[-3ct(1-t+c)+(1-t)(1-t-c)(1-2t+c)\right]$$

This is the equation given in the text.

8) The ratio of profit to copyright income, for any t, is

$$\frac{\pi(t)}{R(t)} = \left(\frac{1-t}{t}\right)\left(\frac{1-t-c}{1-t+c}\right)$$

Let's work this out term by term for the case of

$$t = t^{Ne} = \left(\frac{1}{2}\right)\left(\frac{1-c^2}{2+c+c^2}\right)$$

Firstly, we have $1 - t^{Ne}$ equal to

$$\begin{aligned}
1 - \left(\frac{1}{2}\right)\left(\frac{1-c^2}{2+c+c^2}\right) &= \frac{2(2+c+c^2)-(1-c^2)}{2(2+c+c^2)} \\
&= \frac{4+4c+4c^2-1+c^2}{2(2+c+c^2)} \\
&= \frac{3+4c+5c^2}{2(2+c+c^2)}
\end{aligned}$$

Thus, the ratio of $1 - t^{Ne}$ to t^{Ne} is

$$\frac{1-t}{t} = \frac{3+4c+5c^2}{1-c^2}$$

Second, we need to work out both $1 - t^{Ne} - c$ and $1 - t^{Ne} + c$. The first of these is

$$\begin{aligned}
1 - t^{Ne} - c &= \frac{3+4c+5c^2}{2(2+c+c^2)} - c \\
&= \frac{3+4c+5c^2 - c2(2+c+c^2)}{2(2+c+c^2)} \\
&= \frac{3+4c+5c^2-4c-2c^2-2c^3}{2(2+c+c^2)} \\
&= \frac{3+3c^2-2c^3}{2(2+c+c^2)}
\end{aligned}$$

And the second of them is

$$\begin{aligned}
1 - t^{Ne} + c &= \frac{3+4c+5c^2}{2(2+c+c^2)} + c \\
&= \frac{3+4c+5c^2 + c2(2+c+c^2)}{2(2+c+c^2)} \\
&= \frac{3+4c+5c^2+4c+2c^2+2c^3}{2(2+c+c^2)} \\
&= \frac{3+8c+7c^2+2c^3}{2(2+c+c^2)}
\end{aligned}$$

Therefore, we get

$$\frac{1-t-c}{1-t+c} = \frac{3+3c^2-2c^3}{3+8c+7c^2+2c^3}$$

and so,

$$\frac{\pi}{R} = \left(\frac{3+4c+5c^2}{1-c^2}\right)\left(\frac{3+3c^2-2c^3}{3+8c+7c^2+2c^3}\right)$$

which is the equation given in the text.

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